Olympiad Class Week 5: Cyclic Quadrilaterals

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1 Introduction

Definition: A cyclical quadrilateral is a quadrilateral which can be inscribed in a circle. This simply means that there exists a circle such that each vertex of the quadrilateral lies on the circle's circumference.
Properties: The following facts directly result from the definition of cyclic quadrilateral:
Let $ABCD$ be a cyclic quadrilateral. We have:

$$1)\ \angle ABC + \angle ADC = 180$$

$$2)\ \angle ADC = \frac{ABC}{2}$$

$$3)\ \angle BAC = \angle BDC$$

Furthermore, remember that this worksheet utilizes geometric principles and theorems previously taught and thus when getting stuck, try to recall what was previously taught to see if it's applicable.

2 Problems

1. Prove Ptolemy's Theorem: Let $ABCD$ be a cyclical quadrilateral. Show that $AC \cdot BD = AB \cdot CD + BC \cdot AD$. In other words, the product of the diagonals is equal to the sum of the product of opposite sides.

2. Categorize the following as cyclic or non-cyclic:
(1) A square
(2) A rectangle with unequal lengths
(3) isosceles trapezoid that is not a parallelogram
(4) a parallelogram that is not a rectangle or rhombus
(5) a rhombus that is not a square

3. Let ABC be an equilateral triangle inscribed in a circle. Point D is added on the circle between A and C. What is AD + CD - BD ?

4. Points A, B, C, D lie on a circle. On the chords AB and CD, there exists some points P and Q respectively such that $PQ = 27$. Furthermore, we have $AP = 6$, $PB = 5$, $DQ = 7$, and $QC = 12$. Let X and Y denote the intersection points of the line PQ and the circle. What is XY ?

5. Prove Brahmagupta's Formula: The area of a cyclic quadrilateral is:

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where s denotes the semiperimeter of the quadrilateral, defined as

$$s = \frac{a+b+c+d}{2}$$

You may note the similarity of this formula to that of Heron's for the area of a triangle. You would not be misplaced in this recognition, for Heron's is generalized from Brahmagupta's and Brahmagupta's is generalized by Bretschneider's Formula.

6. Let $ABCD$ be a cyclic quadrilateral. The side lengths of $ABCD$ are distinct integers less than 15 such that $BC \cdot CD = AB \cdot DA$. What is the largest possible value of BD?

7. David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin\varphi_A = \frac{2}{3}$, $\sin\varphi_B = \frac{3}{5}$, and $\sin\varphi_C = \frac{6}{7}$. All three quadrilaterals have the same area K, which can be written in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

8. Let $ABCD$ be a cyclic quadrilateral with $AB = 4, BC = 5, CD = 6$, and $DA = 7$. Let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to line BD, and let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to line AC. The perimeter of $A_1B_1C_1D_1$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.
Note: Although some of you may be familiar with this problem due to its appearance on the 2021 AIME, it is still a very challenging problem.

9. Prove Power of a Point: Let Γ be a circle, and P a point. Let a line through P meet Γ at points A and B, and let another line through P meet Γ at points C and D. Then $PA \cdot PB = PC \cdot PD$. This is a useful tool alongside cyclic quadrilaterals and will likely become a more important technique in the next few weeks.

10. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

11. A triangle is inscribed inside of a circle. Describe where to place a four vertex on this circle to form a cyclic quadrilateral with maximum area.

12. A chord on a circle divides the area into regions A and B. The ratio between the area of region A to the area of region B is 1:35. A rectangle is created with one of its sides being the chord dividing the circle. Johnny begins uniformly throwing darts at the circular dart board has an equal chance of hitting any point on the board. What is the probability that a given dart thrown by Johnny will land in the inscribed rectangle?