ORMC Olympiad Group
Spring: Week 4
Geometry: Triangles III

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Problems

1. **(TJNMO-FR-2018)** Points D and E are chosen on the sides BC and AC of the triangle ABC. $AB = 3, BD = \sqrt{3}, AE = 2, EC = 1$ and $\angle BAD = \angle EDC$. Find ED.

2. $ABCD$ is quadrilateral so that the diagonal AC bisects $\angle DAB$. It is given that $AD = 6, AC = 9, AB = 8$ and $DC/CB = \sqrt{2}$. The side BC can be represented as $\sqrt{\frac{m}{n}}$ where (m, n) are relatively prime positive integers. Find $m \cdot n$

3. Let ABC be a triangle with $BC = 70$ and points M and N are chosen on the sides AB and Ac so that $MN \parallel BC$. Segments CM and BN intersect at the point K. A line which passes through K and parallel to BC intersects with the sides AB and AC at X and Y. Find MN if $XY = 42$.

4. From a point A outside of the circle Γ the tangent AB is drawn, where B is the tangency point. Another line which passes through A cuts the circle Γ at points C and D. If $BC = 5, BD = 7$, what the maximum integer length that the segment AB can take?

5. (**Law of Sin**) ABC is a triangle. Then prove

$$\frac{AB}{\sin C} = \frac{AC}{\sin B} = \frac{BC}{\sin A} = 2R$$

Here R is the radious of the circumcircle.

6. ABC is a triangle and D is any point on the side BC. Let $AB = c, BC = a, CA = b, AD = d, BD = m, DC = n$. Then

$$d^2 = \frac{c^2n + b^2m}{a} - mn$$

7. **Angle Bisector Theorem** ABC is a triangle and D is a point on BC. AD is called *angle bisector* if $\angle BAD = \angle CAD$. Moreover, if AD is an angle bisector, then

$$\frac{BD}{CD} = \frac{AB}{AC}$$

and

$$AD^2 = AB \cdot AC - BD \cdot DC$$

8. **(TNMO-FR 2018 - modified)** ABC is right triangle with hypotenuse AB and it is given that $AC/BC = 3/4$. The interior circle touches sides BC and AC at D and E respectively. AD intersects with the incircle again at the point S. Similarly BE intersects with the incircle again at T. BE and AD intersect at point K.

   (a) Find AS/KD

   (b) Find $(AS/TD)^2$

9. The triangle ABC has sides $AB = 29, AC = \sqrt{409}, BC = 14$. Let M be the midpoint of the side BC. E is chosen on AM such that BE is angel bisector of $\angle MBA$. Find CE.

10. Let ABC be triangle with $AB = 184, AC = 345$ and $\angle BAC = 60°$. Let AD be angel bisector where D is a point on BC. The point X on plane ABC chosen such that $AX \parallel BC$ and $XD \perp AC$. The length of segment AX can be written as $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$ in mod 1000.

11. **(Math Prize for Girls 2014)** Let ABC be a triangle. Points D, E, and F are respectively on the sides BC, CA, and AB of ABC. Suppose that

$$\frac{AE}{AC} = \frac{CD}{CB} = \frac{BF}{BA} = x$$

for some x with $\frac{1}{2} < x < 1$. Segments AD, BE, and CF cut the triangle into 7 nonoverlapping regions: 4 triangles and 3 quadrilaterals. The total area of the 4 triangles equals the total area of the 3 quadrilaterals. Compute the value of x. Express your answer in the form $\frac{k-\sqrt{m}}{n}$, where k and n are positive integers and m is a square-free positive integer.

12. **(AIME 2001II)** Given a triangle, its midpoint triangle is obtained by joining the midpoints of its sides. A sequence of polyhedra P_i is defined recursively as follows: P_0 is a regular tetrahedron whose volume is 1. To obtain P_{i+1}, replace the midpoint triangle of every face of P_i by an outward-pointing regular tetrahedron that has the midpoint triangle as a face. The volume of P_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. **(AIME 2001II)** In quadrilateral $ABCD$, $\angle BAD \cong \angle ADC$ and $\angle ABD \cong \angle BCD$, $AB = 8$, $BD = 10$, and $BC = 6$. The length CD may be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.