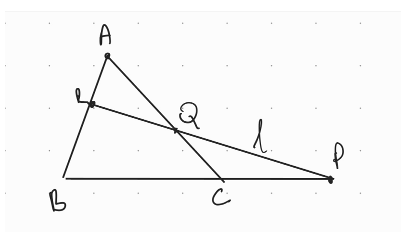
ORMC Olympiad Group
Spring: Week 3
Geometry: Similarity and Triangles II

Osman Akar

April 17, 2022

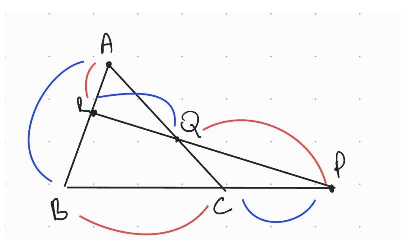
Problems

1. **Menelaus' Theorem** ABC is a triangle. A line l cuts the segments AB and AC at R and Q, and cuts the extension of BC at P.



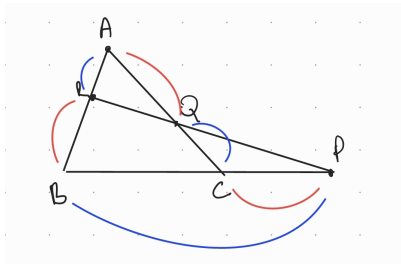
Then

$$\frac{AR}{AB} \cdot \frac{BC}{CP} \cdot \frac{PQ}{QR} = 1$$



and similarly

$$\frac{PC}{PB} \cdot \frac{BR}{RA} \cdot \frac{AQ}{QC} = 1$$



2. **Law of Cos** ABC is a triangle. Then the segment BC can be computed as follow

$$BC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot AC \cdot \cos(\angle BAC)$$

3. ABC is an equilateral triangle with side length 15. Points D, E, F are chosen on the sides BC, CA and AB respectively so that $BD = CE = AF = 7$. When we draw AD, BE, CF, we create a smaller equilateral triangle in the middle of ABC, say that $\triangle XYZ$. Side lenght of $\triangle XYZ$ can be represented as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.

4. **Angle Bisector Theorem** ABC is a triangle and D is a point on BC. AD is called *angle bisector* if $\angle BAD = \angle CAD$. Moreover, if AD is an angle bisector, then

$$\frac{BD}{CD} = \frac{AB}{AC}$$

and

$$AD^2 = AB \cdot AC - BD \cdot DC$$

5. **(Prasolov 1.19)** A straight line passing through vertex A of square $ABCD$ intersects side CD at E and line BC at F. Prove that

$$\frac{1}{AE^2} + \frac{1}{AF^2} = \frac{1}{AB^2}$$

6. **(HMMT 2005 Guts)** Five people of different heights are standing in line from shortest to tallest. As it happens, the tops of their heads are

all collinear; also, for any two successive people, the horizontal distance between them equals the height of the shorter person. If the shortest person is 3 feet tall and the tallest person is 7 feet tall, how tall is the middle person, in feet?

7. Let ABC be a triangle with $BC = 36$. Point D is chosen on the side BC so that $DC = 12$. The line AD and the line which passes through C and parallel to AB intersect at the point K. The line AC and the the line which passes through K and parallel to BC intersect at the point L. What is KL?

8. **(TJNMO-FR-2018)** Points D and E are chosen on the sides BC and AC of the triangle ABC. $AB = 3, BD = \sqrt{3}, AE = 2, EC = 1$ and $\angle BAD = \angle EDC$. Find ED.

9. Let ABC be triangle with $AB = 6, AC = 7, BC = 8$, and P is a point on BC with $BP = 3$. Let Q and R be on sides AC and AB so that $PQ \parallel AB$ and $PR \parallel AC$. The are of the parallelogram $AQPR$ can be written as $\frac{p\sqrt{q}}{r}$ where p and r are relatively prime and q is square-free integer. Find $p + q + r$.

10. **(AHSME - 1950)** A rectangle inscribed in a triangle has its base coinciding with the base b of the triangle. If the altitude of the triangle is h, and the altitude x of the rectangle is half the base of the rectangle, then, calculate x in terms of b and h.

11. **Ceva's Theorem** ABC is a triangle and P is an interior point. The cevians AP, BP, CP cuts the sides BC, CA, AB at points at A_1, B_1, C_1 resppectively. Then

$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = 1$$

12. Let ABC be a triangle with $BC = 70$ and points M and N are chosen on the sides AB and Ac so that $MN \parallel BC$. Segments CM and BN intersect at the point K. A line which passes through K and parallel to BC intersects with the sides AB and AC at X and Y. Find MN if $XY = 42$.

13. **(Prasolov 1.13)** In $\triangle ABC$ bisectors AA_1 and BB_1 are drawn. Prove that the distance from any point M of A_1B_1 to line AB is equal to the

sum of distances from M to AC and BC.

14. **(TJNMO-FR 2017-modified)** Point E is chosen in a parallelogram $ABCD$ so that $\angle AEB + \angle DEC = 180°$. Prove that $\angle DAE = \angle DCE$

15. **(Math Prize for Girls 2014)** Let ABC be a triangle. Points D, E, and F are respectively on the sides BC, CA, and AB of ABC. Suppose that
$$\frac{AE}{AC} = \frac{CD}{CB} = \frac{BF}{BA} = x$$
for some x with $\frac{1}{2} < x < 1$. Segments AD, BE, and CF cut the triangle into 7 nonoverlapping regions: 4 triangles and 3 quadrilaterals. The total area of the 4 triangles equals the total area of the 3 quadrilaterals. Compute the value of x. Express your answer in the form $\frac{k-\sqrt{m}}{n}$, where k and n are positive integers and m is a square-free positive integer.

16. **(TNMO-FR 2018 - modified)** ABC is right triangle with hypotenuse AB and it is given that $AC/BC = 3/4$. The interior circle touches sides BC and AC at D and E respectively. AD intersects with the incircle again at the point S. Similarly BE intersects with the incircle again at T. BE and AD intersect at point K.
    (a) Find AS/KD
    (b) Find $(AS/TD)^2$

17. **(AIME 2001II)** Given a triangle, its midpoint triangle is obtained by joining the midpoints of its sides. A sequence of polyhedra P_i is defined recursively as follows: P_0 is a regular tetrahedron whose volume is 1. To obtain P_{i+1}, replace the midpoint triangle of every face of P_i by an outward-pointing regular tetrahedron that has the midpoint triangle as a face. The volume of P_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

18. **(AIME 2001II)** In quadrilateral $ABCD$, $\angle BAD \cong \angle ADC$ and $\angle ABD \cong \angle BCD$, $AB = 8$, $BD = 10$, and $BC = 6$. The length CD may be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.