Graphs

Translated from shashkovs.ru by Nikita

Definition 1.
We say that a graph is given if a finite set of its vertices is given, and for each pair of different vertices it is known whether they are connected by an edge or not (if connected, these vertices are called the ends of this edge). Sometimes it is convenient to depict a graph as a set of points (vertices) on a plane, some pairs of which are connected by lines (edges). There are graphs with multiple edges (a pair of vertices can be connected by several edges) and loops (a vertex can be connected to itself).

Examples:
a) acquaintance graph: vertices - schoolchildren, edges - acquaintances;
b) map: vertices - countries, edges - pairs of countries with a common border section;
c) cities and roads;
d) graph of the king (knight, rook, queen...): vertices are cells, edges are pairs of cells connected by one move of the king (knight, rook, queen...).

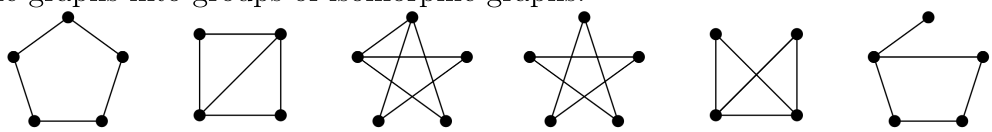
Problem 1.
Draw the graph from example b) for South American countries

Problem 2.
Seven schoolchildren participated in the chess tournament. It is known that Misha played 6 games, Kolya - 5, Ilya and Grisha - three each, Andrey and Seva - two each, and Maxim - one. No two kids played each other twice. Who did Ilya play with?

Problem 3.
Write the digits 0, …, 9 in a row so that the number formed by any 2 following digits is divisible by 7 or 13.

Definition 2.
Two graphs (without loops and multiple edges) are called isomorphic ("identical"), if it is possible to enumerate the vertices of each graph with the same set of different numbers so that the condition is fulfilled: if two vertices in the first graph are connected by an edge, then in the second graph the vertices with the same numbers are connected by an edge, and vice versa.

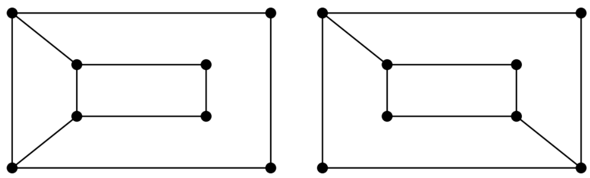
Problem 4.
Divide the graphs into groups of isomorphic graphs.



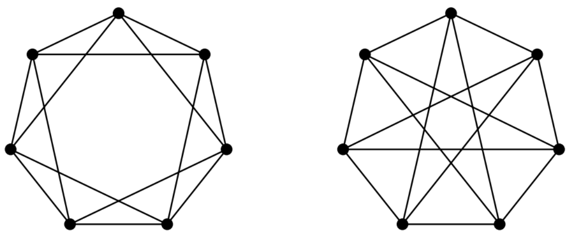
Problem 5.
Draw all mutually non-isomorphic graphs with at most four vertices.

Problem 6.
Are graphs isomorphic?

(a)



(b)



Definition 3.
A complete graph is a graph in which every two different vertices are connected by one edge.

Problem 7.
(a) How many edges can there be in a complete graph with n vertices? (Such a graph is denoted by K_n.)

(b) Draw K_4, K_5, K_6.

Problem 8.
(a) Petya is standing, and 5 people are coming towards him. Prove that among them there are either three who know Petya or three who don't know Petya.

(b) Prove that among any 6 people there are either 3 pairwise acquaintances or 3 pairwise strangers.

(c) What if there are only 5 people?

Problem 9.
A sheet contains **(a)** 178; **(b)** 179 points. Two players play a game: each in his turn connects two points with a line. You cannot connect a pair of points that are already connected. The player loses if after their move from any point you can go to any other along the lines. Who can secure victory?

Definition 4.
The degree $\deg V$ of a vertex V is the number of edges going out of it (for graphs with loops, the loops are counted twice).

Problem 10.
How many edges are in the graph of the **(a)** rook;

(b) king (on the 8×8 board, see the definition of rook graph before Problem 1).

Problem 11.
(a) How are the sum of vertex degrees of a graph and the number of its edges related?

(b) Is it true that the number of vertices of odd degree in any graph is even?

Problem 12.
There are 27 children in a class.
(a) Prove that at least two of them have equally many friends in the class.

(b) Student Yang noticed: the other 26 have a different number of friends in the class. How many of them are friends with Yang?

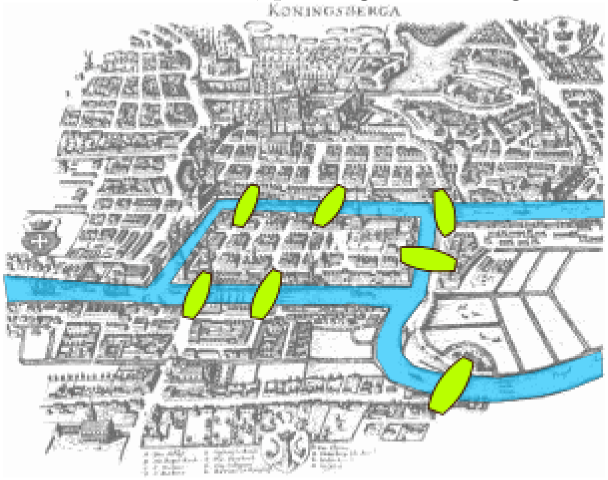
Definition 5.
A path (of length n) is a sequence of vertices and edges connecting adjacent vertices: V_1, e_1, V_2, e_2, $\ldots$, e_n, V_{n+1}. If $V_1 = V_{n+1}$, the path is called cyclic, and if the edges are distinct, it is called a cycle. A path without repeating vertices is called a simple path.

Problem 13.
A tourist arrived at a station and went for a walk along the streets. Prove that he can return to the station at any time, passing only those sections of streets that he has already passed an odd number of times.

Definition 6.
A graph is called connected if every two of its vertices are connected by a path.

Problem 14.
Let G be a graph with n vertices. Suppose that the degree of each vertex is at least $\frac{n-1}{2}$. Prove that it is connected.

Problem 15.
The city of Königsberg in Prussia was set on both sides of the Pregel River, and included two large islands—Kneiphof and Lomse—which were connected to each other, or to the two mainland portions of the city, by seven bridges (see figure on the next page). Was it possible to walk there, passing each bridge exactly once?



Problem 16 (Eulerian graphs)**.**
Let G be a connected graph such that the degree of any of its vertices is even. Prove that
(a) the graph contains a simple cycle;

(b) the edges of the graph can be divided into sets of cycles;

(c) Show that the condition in part (b) is equivalent to the fact that the graph has a cycle containing all edges (one can walk along the edges of the graph in such a way that every edge is used exactly once).