Bounded Sets

Definition. We say that a set of numbers is *bounded* if there is a number M so that the size of every element in the set is no more than M, and *unbounded* if there is no such number M. (Such a number M is called a *bound* on the set.

We say that a set is *bounded above* if there is a number M (an *upper bound* so that every element in the set is no more than M, and *bounded below* if there is a number M (a *lower bound*) so that every element in the set is no less than M.

For each of the following sets, decide whether it is bounded, and if not whether it is still bounded below or bounded above, finding an appropriate bound in each case:

1. The set of all real numbers $\mathbf{R}$.
2. The set of rational numbers $\mathbf{Q}$.
3. The set of all natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$
4. The set of all integers $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$
5. The empty set $\varnothing$, which does not contain any numbers. [1]
6. The set of positive integers with an odd number of divisors.
7. The set of numbers which are a root of a polynomial with integer coefficients.
8. The set of numbers of the form $\frac{n}{2n+1}$, where $n \geq 1$ is an integer. For short, we can specify this set by writing $\{\frac{n}{2n+1} \mid n \in \mathbf{Z}, n \geq 1\}$.
9. The set $\{\frac{1}{n} \mid 0 \neq n \in \mathbf{N}\}$.
10. The set $\{\frac{n^2+1}{50n-7} \mid 1 \leq n \in \mathbf{N}\}$.
11. The set of numbers which are the distance from some point in the plane to the nearest integer lattice point (m, n), where $m, n \in \mathbf{Z}$.
12. The set of possible distances between: a point on the circle $x^2 + y^2 = 1$ and a point on the circle $x^2 + y^2 = 100^2$.
13. The set of possible distances between: a point on the curve $y = \frac{1}{x}$ and a point on the curve $y = -\frac{1}{x}$
14. The set $\{n \in \mathbf{Z} \mid 1000^n > n!\}$
15. The set of numbers of the form $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$, where $n \geq 1$ is an integer.[2]

[1] Note that in mathematics if a true/false statement is made about "all turkeys with five legs", and there aren't any turkeys with five legs, then we say the statement is true ("vacuously true").

[2] Try to imitate the techniques on the next page.

Two Examples

Proposition 1. *The set of numbers of the form* $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$*, where* $n \geq 1$ *is an integer, form an unbounded set.*

Proof. Let $n = 2^k$. Then we have

$$\begin{aligned}
&1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \\
&\quad = 1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots+\left(\frac{1}{2^{k-1}+1}+\frac{1}{2^{k-1}+2}+\ldots+\frac{1}{2^k}\right) \\
&\quad \geq 1+\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2} \\
&\quad = 1+\frac{k}{2}.
\end{aligned}$$

There can not be any number M which is greater than $1+\frac{k}{2}$ for all values of k because if $k = 2\lceil M \rceil$ is the smallest integer larger than $2M$, then

$$1+\frac{k}{2} \geq 1+\frac{2M}{2} = 1+M,$$

which is a contradiction. □

Proposition 2. *The set of numbers of the form* $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$*, where* $n \geq 1$ *is an integer, form a bounded set.*

Proof. If we let

$$S_n = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$$

then

$$\frac{1}{2}S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^{n+1}},$$

so

$$S_n - \frac{1}{2}S_n = \left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^{n+1}}\right),$$

hence

$$\begin{aligned}
\frac{1}{2}S_n &= 1-\frac{1}{2^{n+1}} \\
S_n &= 2\left(1-\frac{1}{2^{n+1}}\right) \leq 2.
\end{aligned}$$

Therefore $M = 2$ is an upper bound on the given set, while all the numbers are positive, so 0 is a lower bound. □

Least Upper Bounds

The following is an important notion in understanding the standard system of real numbers: If S is a set of numbers which is bounded above, then we say that b is a *least upper bound* or *supremum* for S, written $b = \sup S$, if it satisfies two properties

(A) For all $x \in S$, $x \leq b$.

(B) If b' is any other number with the previous property, then $b \leq b'$.

Informally, we may summarize these by saying that

(A) b is an upper bound on the set S.

(B) Of all upper bounds on S, it is "the least".

The notion of a *greatest lower bound*, or *infimum* is defined similarly, and we write $\ell = \inf S$ if ℓ is a greatest lower bound on the set S.

By convention, if $A = \varnothing$ then we say $\sup A = -\infty$, $\inf A = \infty$, if A is not bounded above then we say $\sup A = \infty$, and if A is not bounded below we say $\inf A = -\infty$.

The Dedekind Completeness Axiom

Every nonempty set which is bounded above has a least upper bound.

The Dedekind Completeness Axiom may be taken as a fundamental property of the standard real number system.

Exercises

1. Show that for any set $A \subseteq \mathbf{R}$, $\sup A$ is unique. In other words, if b, b' are two real numbers satisfying the two conditions above, then $b = b'$. (Thus it makes sense to say "the" sup of A).

2. Find, with proof, the least upper bound of the set in the second example on the previous page.

3. Find, with proof, but without using calculus, the value of

$$\inf\{x + \frac{1}{x} \mid x > 0\}.$$

   (Hint: Use the AM-GM inequality: $ab \leq \frac{1}{2}(a^2 + b^2)$.)

4. If $A, B \subseteq \mathbf{R}$ we define their sum set as the set of all possible sums of one element of each set:

$$A + B = \{a + b \mid a \in A, b \in B\}$$

For example $[0, 1] + \{-1, 1\} = [-1, 0] \cup [1, 2]$.

   (a) If A is the rationals and B is the irrationals, what is $A + A$? What is $A + B$? What is $B + B$?

   (b) Prove that if $\sup A$ and $\sup B$ are finite, $\sup(A + B) = \sup A + \sup B$.

5. Given a nonempty set A which is bounded above, define $-A = \{-a \mid a \in A\}$. What is $\inf(-A)$?

6. Assuming $\sup A$ exists, prove that for all $\varepsilon > 0$ there exists $a \in A$ such that $a > \sup A - \varepsilon$.

7. Prove that for any $a > 0$, $\inf\{a^x \mid x \in \mathbf{R}\} = 0$. You may use any standard properties of exponentiation.

8. (a) Assuming the Dedekind Completeness Axiom, prove the *Archimedean property* of $\mathbf{R}$: For any $\varepsilon > 0$ there exists a positive integer n such that $n\varepsilon > 1$. (Hint: Argue by contradiction, considering the supremum of an appropriate set.)

   (b) Use part (a) to prove that for any $x \in \mathbf{R}$ there is an integer n such that $n > x$.