Combinatorics 1

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Problem 1.

(a) List all the anagrams of the word ANna and break them into groups, so that in each group all words are the same, if you do not distinguish between letter sizes.

(b) Divide all sequences of 3 blue and 3 red beads into groups so that for all sequences in each group you get the same necklace, if you turn the sequence into a ring. Note that necklaces can be both rotated and flipped.

(c) How to count all anagrams of the word "anna"?

(d) How to count the number of necklaces with 3 blue and 3 red beads?

Problem 2.

There are 20 people in the class. In how many ways can one choose the chair and the vice chair among them? Or just two co-chairs?

Problem 3.

How many different words (not necessarily meaningful words) can be obtained by rearranging the letters in words:

(a) NET;

(b) TENET;

(c) TENETS;

(d) $\underbrace{AA\ldots A}_{n\ times}\underbrace{BB\ldots B}_{m\ times}$

(e) $\underbrace{c_1c_1\ldots c_1}_{k_1\ times}\underbrace{c_2c_2\ldots c_2}_{k_2\ times}\ldots\underbrace{c_nc_n\ldots c_n}_{k_n\ times}$?

Problem 4.

(a) How many ways are there to choose three attendants in a class of 20 students?

(b) And in how many ways can the chair, his assistant and three attendants be chosen?

Definition 1.

The number of combinations of n things taken k at a time without repetition is called a *binomial coefficient*, and is denotes $\binom{n}{k}$ (read "n choose k").

Problem 5.

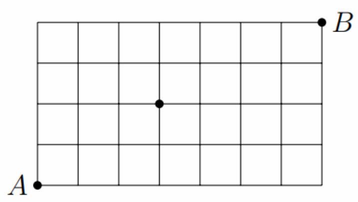
Prove that **(a)** $\binom{n}{k} = \binom{n}{n-k}$;

(b) $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

Problem 6.
Find a closed form expression for $\binom{n}{k}$.

Problem 7.
The figure shows a city plan. The streets have one-way traffic: you can only go "up" or "to the right". **(a)** How many different routes lead from point A to point B?

(b) How many of them goes through the marked intersection?



Problem 8.
In how many ways can a class be seated if there are 27 people and 30 seats?

Problem 9.
In how many ways can 3 pears and 4 apple trees be planted in a row?

Definition 2.
Pascal's triangle is the numerical triangle shown in the figure (there are ones along the edges of the triangle, and each of the other numbers is equal to the sum of the two to the right and left above it).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Problem 10.
Write next 3 strings.

Problem 11.
Prove that the k-th number of the n-th line is equal to $\binom{n}{k}$ (the lines are numbered from top to bottom, starting from zero, and the numbers in the lines are numbered from left to right, also starting from zero).

Problem 12.
Prove that the sum of the numbers in the n-th row of Pascal's triangle is 2^n.

Problem 13.
Prove the identity $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$.

Problem 14.
(a) Expand $(a+b)^2$, $(a+b)^3$ and $(a+b)^4$.

(b) [Newton's binomial theorem] Expand $(a+b)^n$. Prove that each term has a form Ca^kb^{n-k}, where $C = \binom{n}{k}$.

(c) Find the coefficients in x^{17} and x^{18} in the expression $(1+x^5+x^7)^{20}$.

Problem 15.
Prove the identity

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0.$$

Problem 16.
Let's take any number C in Pascal's triangle and add up all the numbers, starting from it and going straight up and to the right. Prove that the sum is equal to the number under C on the right.

Problem 17.
Using the previous problem find **(a)** $T_n = 1+2+3+\cdots+n$;

(b) $\Pi_n = T_1 + T_2 + T_3 + \cdots + T_n$;

(c) $\Pi_1 + \Pi_2 + \Pi_3 + \cdots + \Pi_n$.

Problem 18.
How to use the previous problem to obtain the formulas for the sums $1^2+2^2+3^2+\cdots+k^2$, $1^3+2^3+3^3+\cdots+k^3$, $\ldots$?

Problem 19.
Mark the even numbers in Pascal's triangle. Which lines contains all odd numbers?

Problem 20.
Prove that

$$\binom{0}{p}\binom{m}{q} + \binom{1}{p}\binom{m-1}{q} + \binom{2}{p}\binom{m-2}{q} + \cdots + \binom{m}{p}\binom{0}{q} = \binom{m}{p+q}.$$