Game Theory II - Mixed Strategies and Nash equilibria

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1 Non Zero-Sum Games

Last time, we studied two-player zero-sum games. In general, a game may have more than two players and may not be zero-sum. We will start with two-player non-zero sum games. The first example from the last worksheet, the Prisoner's Dilemma, is an example of a two-player non-zero sum game. Since this is 2-player, it can be represented by a *payoff table* similarly to two-player zero-sum games. The difference now is that the squares are filled with ordered pairs (a, b) where a is Player 1's score and b is Player 2's score - since they no longer have to add up to zero, we need both numbers. The Prisoner's Dilemma has the following payoff table:

	Sp	St
Sp	(5, 5)	(0, 9)
St	(9, 0)	(0, 0)

Similarly to the zero-sum case examined last week, we can also define

Definition 1 *An outcome of a game is a* ***Nash equilibrium*** *if no player could improve their payoff by changing strategies while the other players keep their same strategy.*

Problem 1 *Find all Nash equilibria of the Prisoner's Dilemma.*

Solution: The one-steal and two-steal strategies are all Nash equilibria.

- *Are there Nash equilibria where either player is not playing a maximin strategy?*

  Solution: No

- *Do all Nash equilibria have to give the same scores?*

  Solution: No

Problem 2 *Prove that an outcome of a two-player game is a Nash equilibrium if and only if it corresponds to a square in the payoff table where the first number is the maximum of its column and the second number is the maximum of its row.*

Solution: The first number in a square is the maximum of its column if and only if Player 1 cannot profitably switch strategies while Player 2 keeps the same strategy, and likewise for rows.

As a reminder,

Definition 2 *One strategy **dominates** another strategy for the same player if playing the first strategy will always give that player at least as much ($\geq$) payoff as the second, regardless of the other player's strategy.*

*A strategy **strictly dominates** another strategy for the same player if playing the first strategy will always give that player more ($>$) payoff as the second, regardless of the other player's strategy.*

Problem 3 *Can a Nash equilibrium possibly involve a dominated strategy for either player? What about a strictly dominated strategy for either player? Use your answer to quickly find a Nash equilibrium for the following game, and prove that it's the only one.*

	A	B	C	D	E
1	(5, 5)	(4, 4)	(3, 3)	(2, 2)	(1, 1)
2	(4, 4)	(3, 3)	(2, 2)	(1, 1)	(0, 0)
3	(3, 3)	(2, 2)	(1, 1)	(0, 0)	(-1, -1)
4	(2, 2)	(1, 1)	(0, 0)	(-1, -1)	(-2, -2)
5	(1, 1)	(0, 0)	(-1, -1)	(-2, -2)	(-3, -3)

Solution: A Nash equilibrium may involve a dominated strategy, but never a strictly dominated one, because in that case the player playing the strictly dominated strategy will strictly profit from switching to the dominating strategy. Therefore we can ignore rows and columns corresponding to strictly dominated strategies for either player, so the only Nash equilibrium will be $(A, 1)$.

One very important example of a two-player non zero-sum game is *poker*. In a game of poker, two players will play for a pot - suppose we say that there is a $100 pot. Player 1 will bet $100, adding that amount to the pot. Player 1 will either think they have a better hand than Player 2, in which case we say she is betting for value, or a worse hand, in which case we say she is bluffing. For the sake of simplicity, let us assume that she can correctly judge if her hand is better. Player 2 has the choice to fold, in which case he gives up and Player 1 wins the pot regardless, or to call, in which case he must match Player 1's bet (adding $100 to the pot), and the player with the better hand will win the pot.

Even though the two players do not act simultaneously, because Player 2 has no information about Player 1's strategy (and vice versa), the game is played as if they do act simultaneously. Therefore we can model it with a payoff table.

(NB: In real-life poker, Player 2 would also have the option to raise. For the sake of simplicity, we have ignored this possibility.)

Problem 4 *Draw a payoff table for the above poker game.*

Solution: We want to write down each player's *net* profit. So if Player 1 puts in $100 and wins $300, her score is 200, not 300.

	C	F
V	(200, -100)	(100, 0)
B	(-100, 200)	(100, 0)

Problem 5 *Using the payoff table, determine if this poker game has a Nash equilibrium.*

Solution: This game has no Nash equilibrium.

2 Mixed Strategies

Definition 3 *If a player has n strategies $S_1, ..., S_n$ in a game, a* ***mixed strategy*** *for that player is a sum $p_1S_1 + ... + p_nS_n$ where $p_1, ..., p_n$ are all nonnegative and $p_1 + ... + p_n = 1$. To play this strategy, play the strategy S_i with probability p_i - the p_i are called the* ***frequencies****.*

A strategy that is not mixed is called a ***pure strategy****.*

The payoff of an outcome involving a mixed strategy is the weighted sum of the payoffs, weighted according to the frequencies p_i.

Problem 6 *Complete the following payoff table for one of the example zero-sum games from last week.*

	L	1/2 L + 1/2 R	R
U	4		-2
1/2 U + 1/2 D			
D	-2		0

Problem 7 *Complete the following payoff table for the poker game defined above.*

	C	1/2 C + 1/2 F	F
V	(200, -100)		(100, 0)
1/2 V + 1/2 B			
B	(-100, 200)		(100, 0)

Definition 4 *An outcome involving mixed strategies is a Nash equilibrium if no player can change their frequencies for a larger payoff, while all other players keep their frequencies the same.*

Let's begin with the zero-sum case.

Problem 8 *The following payoff table represents Rock-Paper-Scissors. Intuitively, what do you think a Nash equilibrium should be? Use the table to prove/disprove your idea.*

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Solution: The Nash equilibrium occurs when both players play $1/3R + 1/3P + 1/3S$ - if any player plays any other frequencies, the other player should play the strategy which beats the first player's highest frequency strategy.

Problem 9 *The following zero-sum game was the other example from last week which did not have a pure Nash equilibrium. Find a mixed Nash equilibrium. (Hint: Player 1 will play some mixed strategy $pU + (1 - p)V$. Find some p such that Player 2 should not switch. Then argue similarly for Player 2.)*

	L	R
U	4	-2
D	-2	0

Solution: Suppose Player 1 plays $pU + (1 - p)D$. Player 2 will always have a preferred strategy between L and R unless $p = 1/4$, in which case both L and R have the same payoff. Similarly, suppose Player 2 plays $qL + (1 - q)R$ - unless $q = 1/4$, Player 1 has a preferred strategy. Therefore the Nash equilibrium is given by $(1/4U + 3/4D, 1/4L + 3/4R)$.

Problem 10 *Prove that any two-player zero-sum game has a (possibly mixed) Nash equilibrium. (Hint: In general, the payoff matrix will have some four numbers a, b, c, d. Find a formula for the frequencies which give a Nash equilibrium.)*

Solution: Find formulas for p, q such that $pa + (1-p)c = pb + (1-p)d$ and $qa + (1-q)b = qc + (1-q)d$.

We now move to two-player non-zero sum games which have no pure Nash equilibrium. Our first example is the poker game, defined on Page 2.

Problem 11 *Suppose Player* 1 *bets* \$100. *Find a mixed Nash equilibrium. (Hint: Argue similarly to Problem* 8 *for each player.)*

Solution: Suppose Player 1 plays $pV + (1-p)B$. Player 2 will always have a preferred strategy between C and F unless $p = 2/3$, in which case both C and F have the same payoff. Similarly, suppose Player 2 plays $qC + (1-q)F$ - unless $q = 0$, Player 1 will always prefer V, so the equilibrium strategy is $(2/3V + 1/3B, F)$

Problem 12 *Suppose Player* 1 *bets* \$50. *What mixed strategy should she be playing to achieve the Nash equilibrium?*

Solution: Since Player 2 is risking \$50 to potentially win \$200, Player 1 should play $3/4V + 1/4B$.

Problem 13 *Suppose Player* 1 *bets* \$200. *What mixed strategy should she be playing to achieve the Nash equilibrium?*

Solution: Since Player 2 is risking \$200 to potentially win \$500, Player 1 should play $3/5V + 2/5B$.

Problem 14 *Suppose you are playing against a player who you know or suspect is bluffing exactly half the time. Can they be playing an equilibrium strategy? If so, calculate how much they should bet. If not, how would you punish them?*

Solution: Suppose they are betting x and the pot is y. Then by calling you are risking x to win $2x + y$, so if you are winning half the time, this is always worth it for you. Therefore they cannot be playing an equilibrium strategy, and you can punish it by always calling their bets.

As we see from Problem 1, unlike the zero-sum case, in general we may have multiple Nash equilibria which give different scores. For example, in the *Stag Hunt* game, two hunters are choosing whether to attempt to hunt a stag or a hare. A hare will feed one hunter for one day, while a stag will feed both hunters for two days. Any one hunter can successfully hunt a hare, but it takes both hunters together to successfully hunt a stag. The payoff table for this game is as follows:

	S	H
S	(2, 2)	(0, 1)
H	(0, 1)	(1, 1)

Problem 15 *Find all pure Nash equilibria of the Stag Hunt.*

Solution: The pure Nash equilibria are (S, S) and (H, H).

Problem 16 *Find all mixed Nash equilibria of the Stag Hunt.*

Solution: The only mixed Nash equilibrium is $(1/2S + 1/2H, 1/2S + 1/2H)$.

The Stag Hunt game is an example of a *cooperation game* - a game where two players have the same set of choices and have higher payoff if they pick matching choices than if they pick different ones.

Problem 17 *For each of the following games (described in words and with a payoff table), find all pure and all mixed Nash equilibria. Compare these to your answers for the Stag Hunt.*

In the *driving game*, two cars are facing each other and both driving forward. In order to avoid a collision, both drivers will choose to swerve left or right (from their perspective). If they both swerve in the same direction, they avoid collision, but if they pick different directions, they will crash.

	L	R
L	(0, 0)	(-1, -1)
R	(-1, -1)	(0, 0)

In the *arcade game*, two players are in an arcade and must decide which game they want to play together. Player 1 prefers game A because he is better at that game, while Player 2 prefers game B because she is better at that game. However, the games only give prizes to two players playing together, so if the two players choose different games, they both walk away empty-handed.

	A	B
A	(3, 2)	(0, 0)
B	(0, 0)	(2, 3)

3 Bonus Section: Existence of Equilibria

So far, we have been looking for equilibrium strategies in relatively simple games - two players, with two pure strategies. More complicated games may also have Nash equilibria, as shown by the following theorem.

Theorem 1 *(Nash Existence Theorem) Every game with finitely many players where each player has finitely many pure strategies has a (possibly mixed) Nash equilibrium.*

The proof of the theorem in general is quite geometric, so we will not cover it today. Instead, we consider some cases where an equilibrium might *not* exist. We'll consider some games where players have infinitely many choices.

Problem 18 *Player* 1 *and Player* 2 *both pick a positive integer, and the larger integer wins (+1 to that player, -1 to the other). Does this game have a Nash equilibrium? If so, find it. If not, prove it.*

Solution: There is no Nash equilibrium since for every positive integer there exists a larger one, so any mixed strategy will benefit from replacing the minimum pure strategy it uses with a larger one.

Problem 19 *Player* 1 *and Player* 2 *both pick a real number less than or equal to* 5*, and the larger number wins. Does this game have a Nash equilibrium? If so, find it. If not, prove it.*

Solution: There is a Nash equilibrium, where both players pick 5.

Problem 20 *Player* 1 *and Player* 2 *both pick a real number less than* 5*, and the larger number wins. Does this game have a Nash equilibrium? If so, find it. If not, prove it.*

Solution: There is no Nash equilibrium since for every real number $x < 5$ there is a real number $x < y < 5$, so any mixed strategy will benefit from replacing the minimum pure strategy it uses with a larger one.