Euclid's Algorithm

Recall that *Euclid's algorithm* allows one to easily compute the greatest common divisor (gcd) of two numbers. It is based on the following observation: If $a = qb + r$, where $0 \leq r < b$, then $\gcd(a, b) = \gcd(b, r)$.

Examples

1. Since

$$\begin{aligned} 30 &= 7 \cdot 4 + 2 \\ 6 &= 2 \cdot 3 + 1 \end{aligned}$$

we have $\gcd(30, 7) = \gcd(7, 2) = \gcd(2, 1) = 1$.

2. Since

$$\begin{aligned} 7854 &= 2860 \cdot 2 + 2134 \\ 2860 &= 2134 \cdot 1 + 726 \\ 2134 &= 726 \cdot 2 + 682 \\ 726 &= 682 \cdot 1 + 44 \\ 682 &= 44 \cdot 15 + 22 \\ 44 &= 22 \cdot 2 + 0 \end{aligned}$$

we have

$$\begin{aligned} \gcd(7854, 2860) &= \gcd(2860, 2134) \\ &= \gcd(2134, 726) \\ &= \gcd, 726, 682) \\ &= \gcd(682, 44) \\ &= \gcd(44, 22) \\ &= 22. \end{aligned}$$

Exercises

Use Euclid's algorithm to find each gcd.

1. gcd(210,70) 2. gcd(1050,165) 3. gcd(220,296) 4. gcd(391,51) 5. gcd(225,315)
6. gcd(435,392) 7. gcd(3827,2002) 8. gcd(2012,1984) 9. gcd(525,182) 10. gcd(2431,385)

Long Division of Polynomials

An algorithm analogous to the standard digit by digit algorithm applies to dividing polynomials. To divide $x - 1$ into $x^2 - 4x + 4$, we first ask how many times x goes into x^2 ("it goes in x times"). Then we subtract x times $x - 1$ from $x^2 - 4x + 4$ to obtain $-3x + 4$, and ask how many times x goes into $-3x$ ("it goes in -3 times"). Then subtract -3 times $x - 1$ from $-3x + 4$ to get 1. This is our remainder:

All in all, we get

$$\frac{x^2 - 4x + 4}{x - 1} = x - 3 + \frac{1}{x - 1}.$$

Exercises

Perform each of the following division problems:

1. $x - 2$ divided by $x - 1$
2. $x^2 - 2x + 1$ divided by x
3. $x^3 - x^2 + x - 1$ divided by $x - 1$
4. $x^4 - 3x^3 + 3x^2 - 3x + 2$ divided by $x - 1$
5. $x^4 - 10x^3 + 26x^2 - 50x + 105$ divided by $x^2 + 5$
6. $x^3 - 6x^2 + 11x - 6$ divided by $x - 4$
7. $x^5 - 5x^3 + 4x^2 + x$ divided by $x^2 + 3x + 1$
8. $x^4 - 3x^2 + 3x + 2$ divided by $x^2 - 1$

Polynomial Euclid's Algorithm

With polynomial long division in hand, Euclid's algorithm extends to polynomials! Compute each of the following polynomial gcd's:

1. $\gcd(x^2 - 4, x + 2)$
2. $\gcd(x^2 + 2, x^2 - 2)$
3. $\gcd(x^3 - 1, x^2 - 1)$
4. $\gcd(x^4 - 5x^3 + 8x^2 - 10x + 12, x^4 + x^2 - 2)$
5. $\gcd(x^4 - 16, x^4 + 8x^3 + 24x^2 + 32x + 16)$
6. $\gcd(x^5 + x^4 + x^3 + x^2 + x + 1, x^3 + x^2 + x + 1)$
7. $\gcd(x^7 + x^4 + x^2 + x^2 + x + 1, x^5 + x^4 + x^3 + x^2 + x + 1)$
8. $\gcd(x^4 + 2x^3 - 2x^2 + 2x - 3, x^4 + x^3 + 2x^2 + x + 1)$

More Intersection Number Exercises

Recall that the intersection number of aglebraic curves $f(x, y) = 0$, $g(x, y) = 0$at the origin O is the unique number $I_O(f, g)$ which satisfies the properties:

1. $I_O(f, g)$ is a nonnegative integer or ∞.
2. $I_O(f, g) = I_O(g, f)$
3. $I_O(f, g) \geq 1$ if and only if f and g both contain the origin.
4. $I_O(x, y) = 1$
5. $I_O(f, g) = I_O(f, g + fh)$
6. $I_O(f, gh) = I_O(f, g) + I_O(f, h)$

How many times does each pair of curves intersect at the origin?

1. $y = 0$ and $x = 0$
2. $y = 0$ and $x = 1$
3. $y = x$ and $y = -x$
4. $y = x - 2$ and $x = y + 7$
5. $y = x^2$ and $x = 0$
6. $x^2 + y^2 = 0$ and $x = y$
7. $y^2 = x^3$ and $y - x^2 = 0$
8. $y^2 = x$ and $x = y^2$
9. $y^5 = x^7$ and $y^2 = x^3$
10. $xy^4 + y^3 = x^2$ and $y^5 + x^2 = xy$