Los Angeles Math Circle

by Oleg Gleizer and Olga Radko
with edits from Aaron Anderson and Harris Khan

Euler's number

The goal of this mini-course is to give an accurate construction of the Euler's number e, one of the most fundamental constants in mathematics, physics, economics, and finance. The construction and the applications of the number e to finance and probability are broken into steps and presented as series of problems for students to solve. Harder problems are marked with the red pepper sign.

1 Compounded interest

Let P be the primary capital invested at a constant rate r compounded annually. Let $V(t)$ be the value of the investment in t years.

Problem 1 *Derive the formula for* $V(t)$.

Problem 2 *Derive the formula for* $V(t)$ *if the annual rate* r *is compounded monthly.*

Problem 3 *Derive the formula for* $V(t)$ *if the annual rate* r *is compounded* n *times a year,* $n \in \mathbb{N}$.

In this worksheet, we will explore what it means to compound interest continuously.

2 Preliminaries on Limits

Let $a_1, a_2, a_3, \ldots$, or alternately, $(a_n)_{n=1}^{\infty}$, be a sequence of real numbers. The number $u \in \mathbb{R}$ is called an *upper bound* of $(a_n)_{n=1}^{\infty}$, if $a_n \leq u$ for any $n \in \mathbb{N}$. A sequence $(a_n)_{n=1}^{\infty}$ having an upper bound is called *bounded from above.*

A sequence of real numbers $(a_n)_{n=1}^{\infty}$ is called *monotonically increasing* if $m < n \Rightarrow a_m < a_n$.

Lemma 1 *A monotonically increasing sequence of real numbers bounded from above has a unique limit.*

Intuitively, a limit of a sequence is a point that the sequence gets closer and closer to. For a few examples of monotonically increasing sequences of real numbers, bounded from above, and their limits, consider the sequence $-\frac{1}{1}, -\frac{1}{2}, -\frac{1}{3}, \ldots$, with limit 0, and the sequence $3, 3.1, 3.14, 3.141, \ldots$ which limits to π.

Problem 4 *Let $a_1 = 2$, and when a_n is defined, define a_{n+1} to be $2 + \sqrt{a_n}$. Show that the sequence $(a_n)_{n=1}^{\infty}$ is monotonically increasing and bounded above. Find its limit.*

Problem 5 *Both assumptions of Lemma 1 are necessary.*

Can you come up with an example of a monotonically increasing sequence of real numbers which does not have a limit? How about a sequence of real numbers which is bounded above but does not have a limit?

Now let's provide a formal definition of the limit of a sequence. If you're stuck trying to prove something with this formal definition, you can just give an informal explanation, and come back to the problem later.

Definition 1 *The number A is the limit of the sequence $(a_n)_{n=1}^{\infty}$ if for every positive real number $\varepsilon > 0$, there's some natural number N such that for all $n \geq N$, $|a_n - A| < \varepsilon$. If a sequence $(a_n)_{n=1}^{\infty}$ has a limit, we denote it* $\lim_{n\to\infty} a_n$.

Problem 6

- *Show that 0 is actually the limit of the sequence $(a_n)_{n=1}^{\infty}$ where $a_n = -\frac{1}{n}$.*
- *Show that π is actually the limit of the sequence $3, 3.14, 3.141, \ldots$, where each term gets one more digit of π.*

Problem 7 *Prove that if a sequence $(a_n)_{n=1}^{\infty}$ has a limit, that limit is unique: If both A and B are limits of $(a_n)_{n=1}^{\infty}$, then show $A = B$.*

3 Defining e

Problem 8 *Recall and prove the binomial identity.*

Problem 9 Ⓢ *Prove that*

$$\left(1+\frac{1}{n}\right)^n < 3-\frac{1}{n} \tag{1}$$

for $n = 3, 4, \ldots$

Problem 9 shows that the sequence

$$e_n = \left(1+\frac{1}{n}\right)^n, \ n = 1, 2, \ldots \tag{2}$$

is bounded from above, $e_n < 3$ for $n \in \mathbb{N}$.

The following very useful statement is known as *Bernoulli inequality.*

$$(1+x)^n \geq 1+nx \text{ for } x \geq -1 \text{ and } n \in \mathbb{N} \tag{3}$$

Problem 10 *Use induction to prove 3.*

Problem 11 Ⓢ *Use Bernoulli inequality to prove that the sequence* e_n *defined by (2) is monotonically increasing.*

Problems 9, 11 and lemma 1 show that the sequence $(e_n)_{n=1}^{\infty}$ has a limit.

$$e \stackrel{\text{def}}{=} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{4}$$

4 Continuously compounding interest

Problem 12 *Derive the formula for* $V(t)$ *if the annual rate* r *is compounded continuously.*

Problem 13 *Which of the investments described in problems 1, 2, and 12 is a better choice? Why?*

5 More about e

Problem 14 *Similarly,*

Prove the following formula.

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = e \tag{5}$$

Problem 15 *Prove the following formula.*

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e \tag{6}$$

Problem 16 Ⓢ *Prove the following formula.*

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e} \tag{7}$$

Problem 17 *Prove the following formula.*

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \tag{8}$$

Note that (7) is a particular case of (8) for $x = -1$.

The following very important formula will be proven in a Calculus course.

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 2 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \tag{9}$$

Problem 18 *Find the first six significant digits of e.*

Problem 19 *If $f : \mathbb{R} \to \mathbb{R}$ is a function, we say that L is a limit of f at $+\infty$ when for every positive real $\varepsilon > 0$, there exists some real number M such that if $x \geq M$, then $|f(x) - L| < \varepsilon$. In that case, we say that $L = \lim_{x\to+\infty} f(x)$.*

Prove the following formula.

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e \tag{10}$$

Hint: if $x > 0$, then $\lfloor x\rfloor \leq x \leq \lceil x\rceil$.

Problem 20 *If* $f : \mathbb{R} \to \mathbb{R}$ *is a function, we say that* L *is a limit of* f *at* 0 *on the right when for every positive real* $\varepsilon > 0$*, there exists some (small) positive real number* $\delta > 0$ *such that if* $0 < x < \delta$*, then* $|f(x) - L| < \varepsilon$*. In that case, we say that* $L = \lim_{x \to 0^+} f(x)$*.*

Prove the following formula.

$$\lim_{x \to 0^+} (1 + x)^{\frac{1}{x}} = e$$

6 Euler's number and probability

Problem 21 *A gambler plays a slot machine* n *times. Each time, his chance to win is* p*. What is his chance to win* k *times?*

Problem 22 *A gambler plays 10,000 times a slot machine that pays out one time in 10,000. What is the chance that the gambler loses every bet?*

Problem 23 *A group of* n *people are participating in a gift exchange. Each person puts their name in a hat, and then everyone draws a random name from the hat. For large* n*, what is the probability that someone draws their own name?*