Game Theory I - Definitions and Basic Examples

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1 Strategies and Domination

In these worksheets, we will study a few kinds of games, and try to mathematically determine what *rational* behavior is in each game, and then determine what the outcome of the game will be if each of the players is rational. The first familiar example of such a game is the *Prisoner's Dilemma*. There are several variations, so let us consider the following:

Players 1 and 2 are playing for a total of \$10, held by another person (the referee, let's say). Both players have the option to split or steal the money, and both players must indicate their decision at the same time to the referee. If both players choose to split, they each get \$5. If one player steals, they get \$9 while the other player gets nothing (the referee keeps the extra cash). Finally, if both players steal, then they both get nothing.

Definition 1 *The options for each player (split and steal) are called **strategies**. One strategy is said to **dominate** another if playing the first strategy will always result in more (meaning $\geq$) points than the second, regardless of what the other player plays.*

Problem 1 *In the Prisoner's Dilemma, which strategies are dominated by which strategies?*

Solution: Split is dominated by steal. If the other player steals, you get nothing regardless, and if the other person splits, stealing is more profitable.

Problem 2 *If both players are playing to maximize their own profit, will either player play a dominated strategy? Assuming that both players are playing to maximize their own profit, predict the result of the Prisoner's Dilemma.*

Solution: Neither player will play a dominated strategy. Therefore both players will steal and neither player gets any money.

Problem 3 *Now suppose both players are playing to maximize their total winnings. Will either player play a dominated strategy in this case?*

Solution: Both players should split, because in either case stealing takes more money out of the game.

Definition 2 *An* ***outcome*** *is a choice of strategy for each player.*

2 Zero-Sum Games

Definition 3 *A* ***zero-sum game*** *is a two-player game where Player* 1 *and Player* 2*'s scores add up to zero in every outcome.*

To analyze zero-sum games, we therefore need better tools. One of these is the *payoff table*, which is a table with rows corresponding to Player 1's strategy, columns corresponding to Player 2's strategy, and squares representing the outcome consisting of those two strategies. In each square, we record Player 1's score in that outcome - since the game is zero-sum, we automatically know that Player 2's score is the negative of that. For example, the following is a payoff table for Rock-Paper-Scissors:

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Problem 4 *Use the payoff table to decide if Rock-Paper-Scissors has a dominated strategy for either player.*

Solution: Comparing rows, we see that a strategy for Player 1 dominates another if the numbers in one row are all at least as large as the corresponding numbers in the other row. So we see that Player 1 does not have a dominated strategy. Doing the same to columns shows that Player 2 also does not have a dominated strategy.

Problem 5 *Can we predict the result of a Rock-Paper-Scissors game, if both players are playing to maximize their own score?*

Solution: No, since each player has multiple non-dominated strategies.

Problem 6 *Given the following payoff tables, decide for each game whether either player has a dominated strategy and whether we can predict the result if both players are playing to maximize their own score.*

	L	R
U	-1	2
D	-1	-2

	L	R
U	4	-2
D	-2	0

Solution: In the first game, Player 1's U strategy dominates their D strategy, so they will always play U. Player 2 should therefore play L since Player 1 will always play U, so Player 2 wins one point.

In the second game, neither player has a dominated strategy. We cannot predict the result because both players have multiple non-dominated strategies.

Definition 4 *In a two-player zero-sum game,*

- *For each strategy of each player, the* ***minimum*** *of that strategy is the minimum score of all the outcomes where that player chooses that strategy. The* ***maximum*** *of that strategy is the maximum score of all the outcomes where that player chooses that strategy.*
- *For each player, that player's* ***maximin*** *is the highest minimum of their strategies. Their strategy that achieves this highest minimum is called a* ***maximin strategy*** *(there may be multiple).*
- *For each player, that player's* ***minimax*** *is the lowest maximum of their strategies. Their strategy that achieves this lowest maximum is called a* ***minimax strategy*** *(there may be multiple).*

Problem 7 *For Rock-Paper-Scissors and both games in Problem 8, find Player* 1*'s maximin and minimax. Which are their maximin and minimax strategies?*

Solution: In Rock-Paper-Scissors, Player 1's maximin is -1 and his minimax is 1. Every strategy is a maximin and a minimax strategy.

In the first game from Problem 8, Player 1's maximin is -1 and his minimax is -1. U is a maximin strategy while D is a minimax strategy.

In the second game from Problem 8, Player 1's maximin is -2 and his minimax is 0. D is a minimax strategy, while both strategies are maximin.

Problem 8 *For Rock-Paper-Scissors and both games in Problem 8, if both players are playing their maximin strategies, can you predict the result of the game?*

Solution: In Rock-Paper-Scissors, both players have multiple maximin strategies, so we cannot predict the result.

In the first game from Problem 8, Player 1 will play U while Player 2 will play L, so Player 2 gets one point.

In the second game from Problem 8, Player 2 will play R, while Player 1 will play either strategy, so we cannot predict the result.

Problem 9 *Show that*

$$\textit{Player 1's maximin} = -\textit{Player 2's minimax}$$
$$\textit{Player 1's minimax} = -\textit{Player 2's maximin}$$

Solution: Use the fact that $-\max(a, b) = \min(-a, -b)$, and vice versa.

3 Nash Equilibriums

Definition 5 *A **saddle point** of a payoff table is a point which is the minimum of its row and the maximum of its column.*

Problem 10 *Of the three games we've seen, which of their tables have saddle points? Where are the saddle points?*

Solution: Only the first game from Problem 8 has a saddle point, which is the top left square.

Problem 11 *Prove that all saddle points give the same score. What score is it?*

Solution: All saddle points must give Player 1's maximin score.

Definition 6 *An outcome of a two-player zero-sum game is a **Nash equilibrium** if neither player could improve their score by changing strategies while the other player keeps their same strategy.*

Problem 12 *Of the three games we've seen, which has a Nash equilibrium? Which outcome is it?*

Solution: Only the first game from Problem 8 has a Nash equilibrium, which is the outcome where Player 1 plays U and Player 2 plays L.

Problem 13 *Prove that an outcome is a Nash equilibrium if and only if both players are playing maximin strategies and both players are achieving their maximin scores.*

Solution: If an outcome is a Nash equilibrium, then the fact that Player 1 cannot profitably switch strategies means that the outcome is a minimum for Player 2's given strategy. The fact that Player 2 cannot profitably switch strategies means that the given outcome is the best minimum they have - their maximin. Therefore Player 2 is playing a maximin strategy and achieving their maximin score. The symmetric argument also shows this for Player 1.

Conversely, suppose that both players are playing maximin strategies and achieving their maximin scores. Then in particular they are both achieving their minimum scores for their strategy, so their opponent cannot profitably switch, and so the outcome is a Nash equilibrium.

Problem 14 *Using the previous problem, show that every Nash equilibrium of a game gives the same score.*

Solution: By the previous problem, every Nash equilibrium has a score equal to Player 1's maximin.

Problem 15 *Suppose a game has a Nash equilibrium. If you played it against someone who was cheating, in the sense that they always know which strategy you pick, what is the best score that you can get?*

Solution: By definition, you can always achieve the Nash equilibrium score, and this is the best score you can achieve, because your opponent can always force you to score the minimum of whichever strategy you pick, so the Nash equilibrium, which is your maximin, is the highest such score.

Problem 16 *Prove that an outcome is a Nash equilibrium if and only if its corresponding square in the payoff table is a saddle point.*

Solution: Saddle points give both player's maximin scores - they give Player 1's by Problem 11 and Player 2's by a symmetric argument.

4 Bonus Problem - Recursive Pirates

Suppose there are ten pirates, ordered in a heirarchy $A > B > ... > J$, where A is (currently) the captain. The pirates have plundered 1000 gold coins and must share them between the crew members. The captain will propose a scheme for dividing the coins, and then the crew will vote. If the majority or exactly half of the pirates vote in favor, then the coins will be split as proposed, while if the majority vote against, then they will mutiny and kill the captain, with the next pirate in the heirarchy becoming the captain and the process repeating again.

Assume that all the pirates are infinitely smart and infinitely ruthless - that is, if any pirate who is not currently captain is able to choose between getting some number of coins versus getting the same number of coins and also killing the captain, they will always choose the latter. We would like to find out the maximum number of points the captain can keep to herself while not being killed.

Problem 17 *What is the answer to the problem if there is* 1 *pirate?* 2*?*

Problem 18 *Try to see a pattern in the small cases if you can. How many coins can the captain keep in this scenario outlined above?*

Problem 19 *Does this pattern change if you have more pirates? Try it with* 100*,* 1000*, and* 5000 *pirates.*