ADVANCED 2 COMPETITION I SPRING 2022

OLGA RADKO MATH CIRCLE
ADVANCED 2
MAY 1, 2022

1. Stage 1

Problem 1.1 (1 point)**.** Find

$$\sum_{i=1}^{n} \log\left(\frac{i+1}{i}\right).$$

in terms of n.

Problem 1.2 (1 point)**.** For how many positive integers n is $n^2 - 3n + 2$ a prime number?

Problem 1.3 (1 point)**.** Let us take a sequence $a_0 = 1$, and $a_n = 1 + \frac{1}{a_{n-1}}$. What does this sequence converge to?

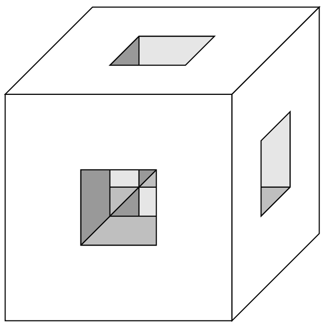
Problem 1.4 (1 point)**.** Say if the following functions are polynomial time or not (all must be correct for points):

(1) 2^n
(2) $2^{\log(n)}$
(3) $2^{\log(n)^2}$
(4) $\log(n^n)$
(5) $\log(n)^{\log(n)}$

Problem 1.5 (1 point)**.** A game is played with tokens according to the following rule. In each round, the player with the most tokens gives one token to each of the other players and also places one token into a discard pile. The game ends when some player runs out of tokens. Players A, B, and C start with 15, 14, and 13 tokens, respectively. How many rounds will there be in this game?

2. Stage 2

Problem 2.1 (2 points)**.** Consider the following figure (a cube with a hole cut through the entire object on each face).



What is the total angular defect of this object?

Problem 2.2 (2 points)**.** Suppose that in some interactive proof, each round of interaction on an F-instance gives a $\frac{1}{n}$ probability of detecting the lie. How many iterations are needed to achieve a 99% chance of detecting the lie? (You can give an approximation if the exact number is hard to compute.)

Problem 2.3 (2 point)**.** Let us again take the sequence $a_n = 1 + \frac{1}{a_{n-1}}$, but instead, we do not specify what a_0 is. Find every value of a_0 that makes this sequence *not* converge to the same value as in Problem 1.3.

Problem 2.4 (2 points)**.** Play the following games three times with a docent. You get full points if you manage to win at least one out of three times.

An integer n, unknown to you, has been randomly chosen in the interval [1, 2002] with uniform probability. Your objective is to select n in an odd number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you must guess an integer on your next turn among the numbers that are still feasibly correct.

Hint: There exists a strategy for which your winning probability for one round is $\frac{2}{3}$.

Problem 2.5 (2 point)**.** Let x, y, z be positive real numbers with $x + 2y + 3z = 1$. Find the maximum value of $\min(2xy, 3xz, 6yz)$.

3. Stage 3

Problem 3.1 (2 points)**.** In the following problem, Aerith and Bob are aware of the situation.The problem begins after this statement.

Cheryl chooses a word in this problem and tells its first letter to Aerith and its last letter to Bob. The following conversation ensues over a series of emails:

Aerith: "I don't know her word, do you?"

Bob: "No, in fact, I don't know if we can ever figure out what her word is without having more information."

Aerith: "Then I do know what it is!"

Bob: "Now I also do."

What is Cheryl's word?

Problem 3.2 (2 points)**.** Recall: a *geometric sequence* is a sequence of the form $a_0 = \ell$, $a_n = \ell \cdot r^n$ for some ℓ, r. Take any three distinct prime numbers (remember, 1 is not prime), and name them p_1, p_2, p_3. How many possible distinct geometric sequences are there that contain p_1, p_2, and p_3 at least once? (The answer is a nonnegative integer or ∞, it does not depend on the primes selected).

Problem 3.3 (2 points)**.** Which of the following games of Nim are winning for the first player? (you must correctly identify all games for any points)
Each list of numbers is a list of piles, each of which starts with the corresponding number of objects. On your turn you may remove any number of objects from one pile (so this is $(\infty, 1)$*-Nim), and a player loses when they cannot remove any objects.*

(1) 1, 1
(2) 1, 2, 3
(3) 2, 4, 7
(4) 8, 8, 8, 8
(5) 4, 7, 9, 13, 41

Problem 3.4 (3 points)**.** "Very Frustrating Game" has six levels. When a level is attempted, the player goes to the next level if they succeed, but back to the previous level if they fail (or if they are on level 1 they restart). "Very average gamer" Swee Hong has a $\frac{1}{2}$ success rate on all levels. How many level attempts on average would it take him to complete the game?

Problem 3.5 (3 points)**.** Let $p_n(k)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Find the value of

$$\sum_{k=0}^{n} k \cdot p_n(k).$$

(Remark: A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if $f(i) = i$.)

4. Stage 4

Problem 4.1 (3 points)**.** Suppose chameleons are either red, green, or blue, and whenever two of them of different colors meet, they both change to the third color. Otherwise, they don't change colors. Suppose you start with 12 red chameleons, 13 green chameleons, and 14 blue chameleons. Is it possible that at some point all the chameleons will be red?

Problem 4.2 (3 points)**.** Without a calculator, find a factor of $85^9 - 21^9 + 6^9$ that is between 2000 and 3000.

Problem 4.3 (3 points)**.** Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers. Find $F(2020, 1981)$.

Problem 4.4 (4 points)**.** Suppose that the integers $1, 2, 3, \ldots, 2020$ are written down in random order. What is the probability that at no time during the process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

Problem 4.5 (4 points)**.** Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find all numbers n for which some player ends up with all n pennies.