# Explorations in Projective Geometry

Recall that $\mathbf{R}^2$ is defined as the set of all ordered pairs $(x, y)$, where $x$ and $y$ are real numbers, while $\mathbf{R}^3$ is defined as the set of all ordered triples $(x, y, z)$, where each coordinate is a real number. The $xy$-plane in $\mathbf{R}^3$ is the set of all points whose $z$ coordinate is 0. It is sometimes convenient to think of the "usual" $xy$-plane, i.e. $\mathbf{R}^2$, as being the same as this set. Similarly we may talk about the $xz$-plane and the $yz$-plane.

Also recall that last time we discussed *perspectivities*, which are mappings from one plane to another determined by using the erspective of one's eye to match up points on the two planes. A composition of perspectivities is called a *projectivity*.

1. Perspective drawings typically only look correct when viewed from exactly the artist's viewpoint. Typically, however, only slight errors result, unless the picture is viewed from an extreme angle. *Anamorphosis* is an artistic style where a drawing *only* looks correct from an extreme angle. Here are some examples:[1]

   Explain why these examples show that the composition of two perspectivities is *not* in general another perspectivity.

[1] John Stillwell, *Mathematics and Its History*, Springer.

2. A projection from one line to another

   (a) Give an explicit example to show that projection from one line to another does not preserve distances.

   (b) Show that any three points on a line can be projected onto any other three points. Thus any property of three points can not be preserved under such a projection.

   (c*) Show that the cross ratio of four points is preserved under a projection.

3. Suppose we place our eye at the point $(0, -4, 4) \in \mathbf{R}^3$.

   (a) If a line passes through our eye and a point on the $xy$-plane with coordinates $(x, y, 0)$, show that it meets the $xz$-plane at a point $(X, 0, Z)$ whose coordinates satisfy

$$X = \frac{4x}{y+4} \qquad Z = \frac{4y'}{y'+4}.$$

   (b) If we begin with the curve $y = x^2$, and project each point onto the $xz$-plane from our perspective, what is the equation of the resulting curve in terms of $X, Z$?

4. If our eye is at the origin (0,0,0), and we consider the curve of points on the set $z = 1$ whose $(x, y)$ coordiantes satisfy the equation $y = x^3$, what equation is satisfied by the $(x, z)$ coordinates of the projected curve on the set where $y = 1$?