THE P VS. NP PROBLEM

GLENN SUN
OLGA RADKO MATH CIRCLE ADVANCED 2

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1 Main Problems

In 2000, the Clay Mathematical Institute put a $1 million prize on 7 important problems in mathematics, called the Millennium Prize Problems. One of them is the P vs. NP question, a question in theoretical computer science: if a problem can be checked quickly by an algorithm, can it also be solved quickly by an algorithm? You might know from experience that it takes a lot longer to solve your homework by yourself than to copy the answers from a friend and just check that the answers are right. Hence, most people believe the answer to the question is no, but where would you even start to try proving this?

There are lots of words that we need to define first: what is a *problem*, what does it mean for an *algorithm* to *solve it* or *check answers*, and how fast do we mean by *quickly*?

An *algorithm* is just a set of unambiguous instructions to solve a problem (think like a robot). For example, consider finding the largest number from a list of numbers. That is, if you are given the list $(9, 9, 222, -10, 3)$, you are supposed to output 222. To have a correct algorithm, your instructions have to work *in general*, no matter what the list is. For example, if we call the input to this problem $(x_1, \ldots, x_n)$, the following could work as an algorithm:

1. Let $m = x_1$.
2. For each x_i in $(x_2, \ldots, x_n)$,
   (a) If $x_i > m$, update m to equal x_i.
   (b) Otherwise, go on to the next element.
3. Output m as the maximum number.

Problem 1. Give an algorithm to solve the following problems. Don't worry about how fast your algorithm is. Trying everything is a valid solution.

1. Given a list of distinct integers $(x_1, \ldots, x_n)$, find the i such that x_i is the largest number in the list.
2. Given two lists of numbers $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, determine if the two lists have a common element.

We will very often look at problems involving graphs. Recall that a graph $G = (V, E)$ is a collection of vertices and edges. When a graph is the input to a problem, you should think that we have for every vertex, a list of the vertices it is directly connected to by an edge.

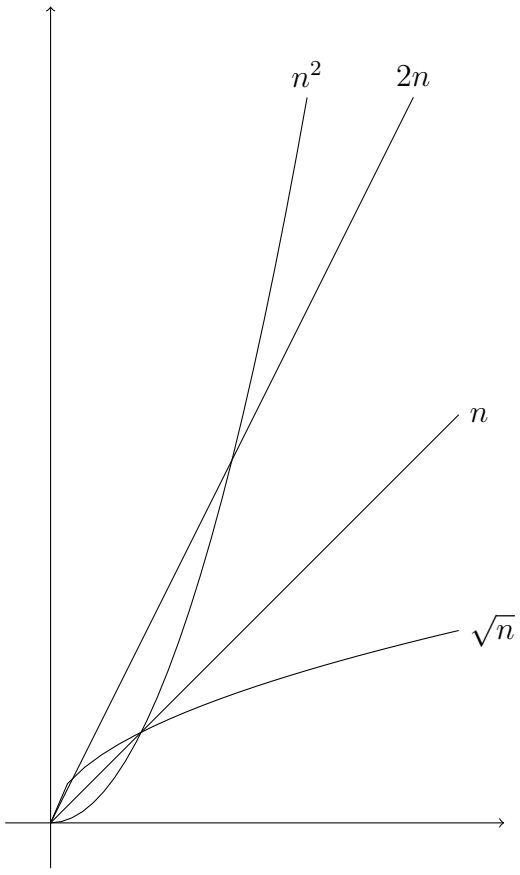
Problem 2. Give an algorithm for the following problems. Don't worry about how fast your algorithm is. Trying everything is a valid solution.

1. A graph is *connected* if there is a way to walk between any two vertices using edges. Given a graph, determine if it is connected.
2. A *coloring* of the graph is a way to assign colors to the vertices such that every edge has different colors on the endpoints. Given a graph, determine if it can be colored using just 3 colors.

Next, we analyze how fast algorithms run.

Definition 1. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be $O(g(n))$ (big-O of $g(n)$) if for large n, $f(n)$ is at most a constant multiple of $g(n)$.

We think of $g(n)$ as a loose upper bound of $f(n)$. In the below picture, $\sqrt{n}$ is $O(n)$ because it is eventually always less than n. Although $2n$ is always greater than n, it is bounded by constant multiple of n, so it is also $O(n)$. The n^2 line is not $O(n)$ because it grows uncontrollably compared to n: for any c, n^2 is eventually greater than cn (when $n > c$).



Problem 3. Are the following functions $O(1)$? $O(n)$? $O(n^2)$? $O(2^n)$? $O(n!)$?

1. $f(n) = 1,000,000$
2. $f(n) = \sqrt{n}$
3. $f(n) = 4n^2 + 2n$
4. $f(n) = 3^n$

We use big-O notation to analyze algorithms because algorithms can get complicated, and we want an easy way to roughly talk about how fast they are without all the constants. For example, in our algorithm to find the largest number in a list,

- Steps 1 and 3 take a constant amount of time to do.
- The inside of step 2 takes at most some constant amount of time to do every iteration.
- The inside of step 2 is repeated n times.

So the total amount of time it takes is at most something like $c_1 + nc_2$, where c_1 and c_2 are some constants. Therefore, we say that this algorithm is $O(n)$.

Problem 4. For each algorithm you wrote in Problems 1 and 2, give a rough upper bound on how many steps are taken on an input of size n using big-O notation. (For graphs, $n = |V|$. Remember that there are at most $\binom{n}{2}$ edges.)

Definition 2. An algorithm runs in *polynomial time* if its running time is $O(n^k)$ for some constant k. (Polynomial time is what we meant by "quickly.")

For example, $n^4 + 3n^2 + 1$ is polynomial time, but 2^n is not polynomial time. (A proof of the second fact requires calculus, so it's in the challenge problems.)

Problem 5. Based on the algorithms you wrote, which of the examples in Problems 1 and 2 run in polynomial time?

Definition 3. A decision problem is informally, a question to which the answer is always yes (T) or no (F). Formally, a decision problem is a function $f : X \to \{\mathsf{T}, \mathsf{F}\}$, where X is the set of all possible inputs. We say that $x \in X$ is a T-instance if $f(x) = \mathsf{T}$ and similarly F-instance if $f(x) = \mathsf{F}$.

An algorithm solves a problem f if for all inputs $x \in X$, the algorithm outputs $f(x)$. The set P is the set of all decision problems that can be solved in polynomial time.

Problem 6. Based on the algorithms you wrote, which of the examples in Problems 1 and 2 belong to P?

The last notion we have to define is what it means to check a solution.

Definition 4. The set NP is the set of all decision problems $f : X \to \{\mathsf{T}, \mathsf{F}\}$ with an polynomial-time algorithm that "checks proofs of $f(x) = \mathsf{T}$," i.e.

- The input consists of both some $x \in X$, and an attempted *proof* y.
- For all T-instances, there exists a proof for which the algorithm outputs T.
- For all F-instances, the algorithm outputs F for all proofs.

An example of a problem in NP is the subset sum problem. It says: given a set of positive integers S and another integer k, determine if there exists a subset $S' \subseteq S$ that you can add up to get exactly k.

Letting S' be the proof, all we need to do to check that S' is a valid proof is to check that $S' \subset S$ and the sum of the elements in S' is k. This is easy to do in polynomial time, and there exists a proof for x if and only if x is a T-instance, as we wanted.

Note that the original question of determining if there exists S' seems to require much more computation — there are 2^n subsets of S (where $n = |S|$), so it is not possible to try them all in polynomial time.

Problem 7. Show that the following problems belong to NP by identifying a proof for every T-instance x, and giving an algorithm to check the proof.

1. The 3-coloring problem from Problem 2.
2. Given a positive integer N that is n digits long, determine if N is composite. (The number of digits is the input length.)

Problem 8. Prove that $\mathsf{P} \subset \mathsf{NP}$.

The P vs. NP problem asks if the reverse inclusion $\mathsf{NP} \subset \mathsf{P}$ is true. That is, are there problems that can be checked in polynomial time, but cannot be solved directly in polynomial time? Most people think this is *false*, but mathematicians and computer scientists still don't know for sure.

Definition 5. A *reduction* from f to g is a way to solve f assuming you know how to solve g.

Problem 9. In this problem, we will give a reduction from 3-coloring to 4-coloring.

1. Let G be a graph. Construct a graph H such that G can be 3-colored if and only if H can be 4-colored. (Hint: add a vertex.)
2. Use the above the give a reduction from 3-coloring to 4-coloring.

Definition 6. A problem g is NP-complete if $g \in \mathsf{NP}$ and all other problems in NP can be reduced to g in polynomial time.

Problem 10.

1. It is a fact that 3-coloring is NP-complete. (See challenge problems for a proof.) Show that 4-coloring is NP-complete.
2. Explain why if g is an NP-complete problem, then to prove that $\mathsf{NP} \subset \mathsf{P}$, it suffices to show that $g \in \mathsf{P}$.

Part 2 above is also the reason why we want to consider polynomial time as our definition of "quickly." Unlike, say, the set of problems that can be computed in $O(n^2)$ time, polynomial time is closed under composition and multiplication, which is very useful.

Problem 11. The partition problem is: Given a set of positive integers S, determine if S can be split into two subsets that have equal sum. Using the fact that subset sum is NP-complete, we will prove that the partition problem is also NP-complete.

1. Prove that the partition problem is in NP.
2. Show that if partition problem can be solved in polynomial time, then the subset sum problem can also be solved in polynomial time. (Hint: Let a denote the sum of the elements in S and consider $S \cup \{a - 2k\}$.)
3. Conclude that the partition problem is NP-complete.

Many practical problems are NP-complete, so this is a very important class of problems. Here are some examples:

- The 3-coloring problem from Problem 2.
- The subset sum problem from the example above.
- Scheduling problems: Given a set of students who each want to take several classes and a limited number of timeslots, decide if the classes can be scheduled to avoid conflicts.
- Traveling salesman problem: Given a map and a list of cities, find the shortest route that visits all the cities. (Decision version: decide if a route of length $\leq k$ exists.)
- Knapsack problem: Given a knapsack with a weight limit and a list of items with various weights and values, determine which items to put in the knapsack to maximize the value. (Decision version: decide if there exists items to get value $\geq k$.)
- Generating crossword puzzles: Given a set of words and a desired crossword grid, determine if the white squares (not the black squares) can be filled such that every maximal horizontal and vertical sequence of white squares is a word from the set.

2 Challenge Problems

Problem 12. Prove that for any positive integer k,

$$\lim_{n\to\infty} \frac{n^k}{2^n} = 0.$$

Conclude that 2^n is not $O(n^k)$ for any constant k.

Problem 13. Come up with a problem that is (provably) not in P. (Hint: use a diagonalization argument, like in the rational numbers worksheet. Ask for more hints if you're stuck.)

Problem 14. In this problem, we will prove from scratch that 3-coloring is NP-complete. This is a very difficult problem, so feel free to ask your instructor for hints.

First, we define the SAT (satisfiability) problem. Let $\wedge$ denote "and," $\vee$ denote "or," and $\neg$ denote "not." A Boolean formula in CNF form looks something like:

$$(x_1 \vee \neg x_2 \wedge x_3) \wedge (\neg x_4 \vee x_1) \wedge (x_5 \vee x_6 \vee x_2 \vee x_3)$$

Formally, it is a $\wedge$ of *clauses*, each of which is the $\vee$ of possibly negated variables. The SAT problem asks: Given a Boolean formula φ, is there an assignment of T/F values to the x_i that makes the formula true?

1. Show that SAT is in NP.
2. Let $f \in \mathsf{NP}$. Assuming that you know how to solve SAT, sketch an argument to show how to solve f. This shows that SAT is NP-complete.
3. 3SAT is the version of SAT where every clause has as most 3 variables. Show that 3SAT is NP-complete by reducing SAT to 3SAT.
4. Show that 3-coloring is NP-complete by reducing 3SAT to 3-coloring.