LAMC Intermediate I & II **November 2, 2014**

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Warm-up

Recall that cryptarithmetic, also know as cryptarithm, alphametics, or word addition, is a math game of figuring out unknown numbers represented by words. Different letters correspond to different digits. Same letters correspond to same digits. The first digit of a number cannot be zero.

Problem 1 *Solve the following cryptarithm, in German.*

$$\begin{array}{r} E\ I\ N\ S \\ E\ I\ N\ S \\ +\ E\ I\ N\ S \\ E\ I\ N\ S \\ \hline V\ I\ E\ R \end{array}$$

Problem 2 *Compute without a calculator.*

$$\frac{74 \times 147 - 73}{73 \times 147 + 74} =$$

$$\frac{244 \times 395 - 151}{244 + 395 \times 243} =$$

$$\frac{423134 \times 846267 - 423133}{423133 \times 846267 + 423134} =$$

Recall that two figures in the (Euclidean) plane are called *similar*, if they have the same shape, but possibly different size.

Problem 3 *Draw a pair of similar hexagons in the space below.*

Problem 4 *One day, Oleg drew two similar hexagons on a paper sheet and cut them out with scissors. Oleg was quite surprised to find out that the larger hexagon never completely covered the smaller one no matter how he moved the figures on the table. Draw a pair of similar hexagons that have this property.*

Back to Boolean algebra

Problem 5 *Prove that in Boolean algebra addition is distributive with respect to multiplication.*

$$A + (B \times C) = (A + B) \times (A + C)$$

A	B	C	$B \times C$	$A + (B \times C)$	$A + B$	$A + C$	$(A + B) \times (A + C)$
0	0	0					
1	0	0					
0	1	0					
0	0	1					
1	1	0					
1	0	1					
0	1	1					
1	1	1					

Negation of composite statements

The two formulas proven in Problems 6 and 8 below are fundamental for understanding the algebra of logic.

Problem 6 *Prove that* $\neg(A+B) = \neg A \times \neg B$.

A	B	$A+B$	$\neg(A+B)$	$\neg A$	$\neg B$	$\neg A \times \neg B$
0	0					
1	0					
0	1					
1	1					

Problem 7 *Negate the following statement.* **My dad likes to watch football or baseball.**

Problem 8 *Prove that* $\neg(A \times B) = \neg A + \neg B$.

A	B	$A \times B$	$\neg(A \times B)$	$\neg A$	$\neg B$	$\neg A + \neg B$
0	0					
1	0					
0	1					
1	1					

Problem 9 *Negate the following statement.* **My dad likes to watch football and baseball.**

The following two formulas generalize the ones proven in Problems 6 and 8.

$$\neg(A_1 + A_2 + \ldots + A_n) = \neg A_1 \times \neg A_2 \times \ldots \times \neg A_n \tag{1}$$

$$\neg(A_1 \times A_2 \times \ldots \times A_n) = \neg A_1 + \neg A_2 + \ldots + \neg A_n \tag{2}$$

Problem 10 *Simplify the following formulas so that they do not contain a negation of a composite statement.*

- $X = \neg(AB) + \neg B$

$X =$

- $Y = \neg(\neg BC + C)$

$Y =$

- $Z = \neg(\neg AC) + B\neg C$

$Z =$

Problem 11 *Given the statements*

A = *Bob is driving to work.*

B = *Bob is shaving.*

form the statement $X = \neg(AB) + \neg B$ *from Problem 10 in plain English and simplify it in the space below.*

Simplifying Boolean expressions

Two expressions of Boolean algebra are called *equivalent* if they are equal as functions – the same inputs produce the same outputs. The latter can be checked by means of the corresponding truth tables. For example, the expressions $A + BC$ and $(A + B)(A + C)$ are equivalent.

Problem 12 *Prove that the expressions* $A + AB$ *and* A *are equivalent without using the truth table.*

Then use the truth table to check your proof.

A	B	AB	$A + AB$
0	0		
1	0		
0	1		
1	1		

In Problem 12, we have proven the following remarkable Boolean algebra equivalence.

$$A + AB = A \tag{3}$$

Here is one more.

Problem 13 *Prove the following formula without using the truth table.*

$$A(A + B) = A \tag{4}$$

To *simplify* a Boolean algebra expression means to find an equivalent expression that

1. contains no negations of composite statements; and

2. has as few simple statements as possible.

The equivalence

$$A(A + \neg(VW + \neg XYZ)) = A \tag{5}$$

is an example of such a simplification.

Problem 14 *If you decide to check formula 5 using a truth table, how many different inputs would you need to consider?*

Problem 15 *Prove formula 5.*

The following two problems present two more very important equivalences.

Problem 16 *Prove the following formula.*

$$AB + A\neg B = A \tag{6}$$

Problem 17 *Prove the following formula.*

$$(A + B)(A + \neg B) = A \tag{7}$$

Problem 18 *Simplify the following Boolean algebra expressions.*

- $\neg(AB) + \neg A =$

- $A + \neg(\neg AB) =$

- $\neg(\neg A \neg B) + \neg A =$

- $\neg(A + \neg(\neg AB)) =$

- $\neg(AB) + \neg ABC =$

The problem continues on the next page.

- $\neg A + \neg(AB + \neg B) =$

- $A\neg BC + A\neg(BC) + ABC + A\neg B =$

The problem you will see below is similar to the logical problems you have solved at the Circle and at various math competitions. Please solve it any way you like. Further we will show you how to solve the problem using the Boolean algebra machinery we have developed.

Problem 19 *The year is 3014. Four kids got to the final tour of GMC8 (Galactic Math Olympiad for 8th graders), Nathan, Michelle, Laura, and Reinhardt. Some knowledgeable LAMC fans discussed their chances to win. One student thought that Nathan would take the first place and Michelle would take the second. Another student thought that Laura would take the silver while Reinhardt would end up the last of the four. The third student thought that Nathan would be second and Reinhardt third. When the results of the competition came out, it turned out that* *each of the LAMC students had made only one of the two predictions correct. Please find the places Nathan, Michelle, Laura, and Reinhardt got at GMC8-3014.*

A Boolean algebra solution to Problem 19.

The following are the simple statements from Problem 19.

- $N_1 =$ *Nathan takes the first place.*
- $M_2 =$ *Michelle takes the second place.*
- $L_2 =$ *Laura takes the second place.*
- $R_4 =$ *Reinhardt takes the fourth place.*
- $N_2 =$ *Nathan takes the second place.*
- $R_3 =$ *Reinhardt takes the third place.*

Let us use the simple statements above to translate the story into the Boolean algebra language. The first fan made a composite statement N_1M_2 that turned out to be false.

$$N_1M_2 = 0$$

The fact that a half of the guess is true means that either $N_1\neg M_2 = 1$ and $\neg N_1M_2 = 0$ or that $N_1\neg M_2 = 0$ and $\neg N_1M_2 = 1$. This can be expressed by means of a single formula.

$$N_1\neg M_2 + \neg N_1M_2 = 1 \tag{8}$$

A similar translation of the other two fans' predictions into the Boolean algebra language gives us the following.

$$L_2\neg R_4 + \neg L_2R_4 = 1 \tag{9}$$

$$N_2 \neg R_3 + \neg N_2 R_3 = 1 \tag{10}$$

Multiplying 8, 9, and 10 brings together all the information we have about the competition.

$$(N_1 \neg M_2 + \neg N_1 M_2)(L_2 \neg R_4 + \neg L_2 R_4)(N_2 \neg R_3 + \neg N_2 R_3) = 1 \tag{11}$$

Let us first find the product of the second and third factors.

$$(L_2 \neg R_4 + \neg L_2 R_4)(N_2 \neg R_3 + \neg N_2 R_3) = 1$$

Opening parentheses gives the following.

$$L_2 \neg R_4 N_2 \neg R_3 + L_2 \neg R_4 \neg N_2 R_3 + \neg L_2 R_4 N_2 \neg R_3 + \neg L_2 R_4 \neg N_2 R_3 = 1$$

Since Laura and Nathan cannot take the second place simultaneously, $L_2 \neg R_4 N_2 \neg R_3 = 0$. Since Reinhardt cannot take the third and fourth place at the same time, $\neg L_2 R_4 \neg N_2 R_3 = 0$. The above sum shortens to just two terms.

$$L_2 \neg R_4 \neg N_2 R_3 + \neg L_2 R_4 N_2 \neg R_3 = 1$$

This way, 11 boils down to the following .

$$(N_1 \neg M_2 + \neg N_1 M_2)(L_2 \neg R_4 \neg N_2 R_3 + \neg L_2 R_4 N_2 \neg R_3) = 1$$

Let us expand. $N_1 \neg M_2 L_2 \neg R_4 \neg N_2 R_3 + N_1 \neg M_2 \neg L_2 R_4 N_2 \neg R_3 + \neg N_1 M_2 L_2 \neg R_4 \neg N_2 R_3 + \neg N_1 M_2 \neg L_2 R_4 N_2 \neg R_3 = 1$ Since $N_1 N_2 = 0$, the second term is equal to zero. Since $M_2 L_2 = 0$, the third term is equal to zero as well. Since $M_2 N_2 = 0$, the same is true for the last term. We end up with the equation

$$N_1 \neg M_2 L_2 \neg R_4 \neg N_2 R_3 = 1$$

that tells us the results of the competition. Nathan takes the first place, Laura the second, Reinhardt the third. Therefore, Michelle takes the fourth place. There are no contradictions: Michelle is not second, Reinhardt is not fourth, and Nathan is not second. We have solved the problem!

Problem 20 *Before the beginning of a school year, teachers get together to form a schedule. The math teacher wants to have her class either first or second. The history teacher wants to have his class either first or third. The English teacher wants to have her class either second or third. Please use Boolean algebra to help the teachers form the schedule. How many different possibilities do they have?*