

## PROOF, PATTERN, CONJECTURE

MATH CIRCLE (BEGINNERS) 04/15/2012

**(1) (*Counting subsets*)** A *set* is a collection of objects (called *elements*), which we often write as a list between curly braces. One set is **{dog, cat}**, another is **{1,2,3,4,5}**, and a third is **{}**. The last one is the empty set, which has no elements. Every set has some number of *subsets*, which are sets that contain only elements in the original set (but not necessarily all of them). **{dog, cat}** has 4 subsets: **{}**, **{dog}**, **{cat}**, and **{dog, cat}**. (Notice we count both the empty set, and the original set, as subsets of the original set.)

(a) How many subsets does the empty set **{}** have?

(b) How many subsets does **{1}** have?

(c) How many subsets does **{1,2}** have?

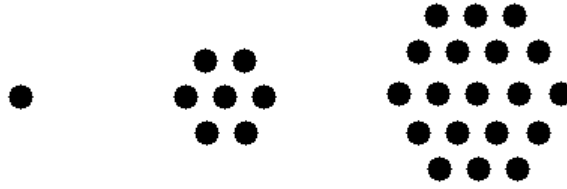
(d) How many subsets of **{1,2,3}**?

(e) How many subsets of **{1,2,3,4}**?

(f) How many subsets of **{1,2,3,4,5}**?

(g) Can you guess how many subsets an  $n$ -element set has, in terms of  $n$ ?

(2) (**Hex Numbers**) Hex numbers are formed by arranging dots in hexagonal shapes with an increasing number of dots per side. The first three hexagonal numbers are  $H_1 = 1$ ,  $H_2 = 7$ , and  $H_3 = 19$ , as demonstrated below:



$$H_1 = 1$$

$$H_2 = 7$$

$$H_3 = 19$$

Draw pictures which represent  $H_4$  and  $H_5$ :

Now compute the following partial sums of the hex number sequence:

$$H_1 =$$

$$H_1 + H_2 =$$

$$H_1 + H_2 + H_3 =$$

$$H_1 + H_2 + H_3 + H_4 =$$

$$H_1 + H_2 + H_3 + H_4 + H_5 =$$

Do you see a pattern? Do you think it continues? How do you know?

**(3) (Goldbach's Conjecture)** The number 4 is a sum of two primes, namely  $4 = 2 + 2$ . Likewise 6 is a sum of two primes,  $6 = 3 + 3$ , and  $8 = 3 + 5$ . Try the next several even numbers—can you write each one as a sum of two primes?

$$10 =$$

$$12 =$$

$$14 =$$

$$16 =$$

$$18 =$$

$$20 =$$

Do you think this pattern continues (is every even number the sum of two primes)?

**(4) (*Euler's Polynomial*)** Consider the expression  $n^2 + n + 41$ . If you plug in zero, you get  $0^2 + 0 + 41 = 41$ . If you plug in 1, you get 43, and if you plug in 2, you get 47. What do you get when you plug in the next several numbers? What do all the results have in common? (Hint: It's more than just that they're all odd...)

$$0^2 + 0 + 41 = 41$$

$$1^2 + 1 + 41 = 43$$

$$2^2 + 2 + 41 = 47$$

$$3^2 + 3 + 41 =$$

$$4^2 + 4 + 41 =$$

$$5^2 + 5 + 41 =$$

$$6^2 + 6 + 41 =$$

$$7^2 + 7 + 41 =$$

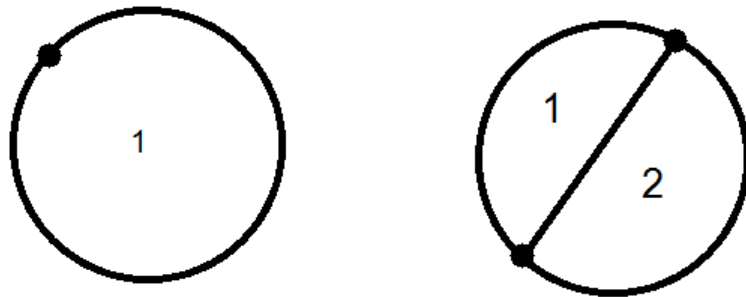
$$8^2 + 8 + 41 =$$

$$9^2 + 9 + 41 =$$

$$10^2 + 10 + 41 =$$

Do you think this pattern continues? How do you know?

(5) (*Circles and chords*) Here are two circles. The first has **one point** on it, and **1 region** inside. The second has **two points** connected by a line segment (or *chord*), with **2 regions** inside:



(a) Draw a circle with **three** points on it, and connect each pair of points by a chord.

How many regions does the circle have?

(b) Draw a circle with **four** points on it, and connect each pair of points by a chord.

How many regions does the circle have?

(c) Draw a circle with **five** points on it, and connect each pair of points by a chord.

How many regions does the circle have?

(d) Can you guess how many regions a circle with  $n$  points on it would have, in terms of  $n$ ?

**(6) (*Odd squares*)** Write down any odd number:

Square it:

Take the remainder on division by 8:

Compare with someone else at your table. Try several different odd numbers. Is there a pattern? Does it always hold?

(7) (**Compare the numbers**) Use a calculator to compare  $\sqrt{\sqrt{9.1}}$  and  $33/19$ . What do you find?

(8) (**Hailstone numbers**) Start with any positive whole number  $n$ , and create a sequence by doing the following process:

- If the current number is even, divide by 2.
- If the current number is odd, multiply by 3 and add 1.

For instance, if we start with **3**, our sequence will be **3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, ...**

Try some examples yourself—for instance, starting with 7.

As you see, once the sequence reaches 4, 2, or 1, it will start to repeat itself. **Question:** Does *every* starting number lead to a sequence which eventually cycles 4, 2, 1, ...? What do you think?

**(9) (*Armen's Identity*)** Write down the first several cube numbers:

$$1^3 = 1$$

$$2^3 = 8$$

$$3^3 =$$

$$4^3 =$$

$$5^3 =$$

$$6^3 =$$

Now consider their partial sums (just as we did with the hex numbers earlier):

$$1^3 = 1$$

$$1^3 + 2^3 =$$

$$1^3 + 2^3 + 3^3 =$$

$$1^3 + 2^3 + 3^3 + 4^3 =$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 =$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 =$$

What kind of numbers are appearing? Is there a deeper pattern? Do you think it continues?