Hyperbolic Space

Aaron Anderson for Olga Radko Math Circle

April 3, 2022

1 Parallel Lines

Let's define two lines to be parallel if they both meet a third line at right angles.

In standard Euclidean geometry, for every line ℓ and point p not on ℓ, there is exactly one line ℓ' through p parallel to ℓ. This is known as *Playfair's axiom*, and it is a version of Euclid's parallel postulate.

Problem 1. Verify that this does *not* hold for your paper hyperbolic surface.

Problem 2. Find a quadrilateral on your surface such that three angles are right (this is called a *Lambert quadrilateral*, but the remaining angle is acute.

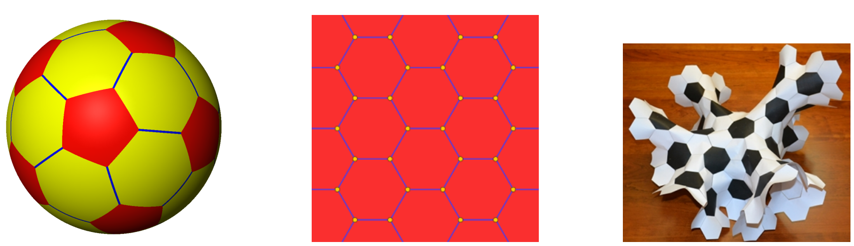
Problem 3. Show that if every triangle's angles add to 180°, every Lambert quadrilateral is a rectangle, and that Playfair's axiom holds.

2 Angle Defect

Definition 1. A *polygonal mesh* is (not-necessarily flat) 2D arrangement of polygons in 3D space, attached by their edges. (Basically, the kind of thing that we just built in our papercraft project).

The *boundary* of a mesh consists of the edges that aren't glued together to other edges, and a mesh is *closed* when it has no boundary (like a cube).

Let's now compare different polygonal meshes, and figure out what the shapes of the polygons tell us about the big-picture geometry of the mesh. In particular, the following three meshes have radically different overall shapes, despite all being made of regular polygons with 5, 6, or 7 sides, mostly hexagons, with 3 meeting at every vertex.



(a) (b) (c)

Figure 1: Polygonal meshes featuring regular hexagons and other regular polygons. Soccer ball, Hextile, Hyperbolic football

Problem 4. In a plane tiling, such as the hextile Figure 1b, what is the sum of the angles of the polygons at a particular vertex?

Problem 5. In Figure 1a and Figure 1c, is the sum of the angles at a vertex more or less than in a plane tiling?

Definition 2. The *angle defect* of a polygonal mesh at a vertex v is $360° - A$, where A is the sum of all angles at v.

Problem 6. What is the sum of the angle defects of the vertices of a cube? What about a tetrahedron, or your other favorite platonic solid? Can you hypothesize a pattern?

Problem 7. If a convex polygon has n sides, what is the sum of its internal angles?

Problem 8. Consider a closed (boundaryless) polygonal mesh with V vertices, E edges, and F faces, each of which is a convex polygon. Find and prove a formula for the sum of the angle defect in terms of V, E, and F.

Problem 9. Calculate the angle defect of a vertex of the hyperbolic football.

Problem 10. Draw some triangles on your hyperbolic football, and measure the total angles. Compare this to the total angle defect of the vertices inside the triangle.

Problem 11. Can you find a general formula relating the total angle defect, V, E, F, and the boundary angles of a general polygonal mesh?