ORMC Olympiad Group Winter 2022
Week 6 Solutions

Sumith Nalabolu

January 21, 2022

Solution 1

We claim that $3p \mid m - 1$.

First, note that $8^p - 1 = (8-1)(8^{p-1} + 8^{p-2} + \ldots + 1)$. Hence $m = 8^{p-1} + 8^{p-2} + \ldots + 1$. This expression contains an odd number of terms since $p > 7$ is prime and therefore odd, and there are p terms in the sum. Furthermore, they alternate between 1 and -1 modulo 3 based on the parity of the exponent, so we see that $m \equiv 1 \pmod 3$. Hence $3 \mid m - 1$.

Also, by Fermat's Little Theorem, and the fact that $\gcd(7, p) = 1$,

$$m \equiv \frac{8^p - 1}{7} \equiv \frac{8-1}{7} \equiv 1 \pmod p.$$

Hence $p \mid m - 1$. Finally, since $\gcd(3, p) = 1$, combining these two facts gives $3p \mid m - 1$. Thus, $m - 1 = 3pk$ for some positive integer k, so

$$2^{m-1} \equiv 2^{3pk} \equiv (8^p)^k \pmod m.$$

Furthermore, by the definition of m,

$$\frac{8^p - 1}{7} \equiv m \equiv 0 \pmod m \implies 8^p \equiv 1 \pmod m$$

from manipulating the equivalence. Hence $(8^p)^k \equiv 1^k \equiv 1 \pmod m$, as desired. □

Solution 2

Due to the definition of remainder, the given conditions imply that $b < a$ and $a - 2 < b$. So, $a - 2 < b < a$, but the only number between $a - 2$ and a is $a - 1$, so this implies $b = a - 1$. Then, this means $n \equiv b \equiv -1 \pmod a$, and $n \equiv a - 2 \equiv -1 \pmod b$. Also, since $\gcd(a, b) = 1$ (since they are consecutive integers), we can combine these to conclude $n \equiv -1 \pmod{ab}$. That is, $n \equiv -1 \pmod{a(a-1)}$, where $a \geq 6$ (since $a - 1 = b \geq 5$).

Now we simply count the answer. Since $n \geq 100$: for $a = 6$, $n = 35, 71$ work; for $a = 7$, $n = 41, 83$ work; for $n = 8$, $n = 55$ works; for $a = 9$, $n = 71$ works; for $a = 10$, $n = 89$ works. So there are $\boxed{7}$ solutions. □

Solution 3

Note that $a^3 - b^3 = (a-b)(a^2 + ab + b^2$, so the given condition implies $p \mid a^3 - b^3$. That is, $a^3 \equiv b^3 \pmod p$. Also, by Fermat's Little Theorem, $a^{3k+1} \equiv 1 \equiv b^{3k+1} \pmod p$. From the first derived fact, we also know $a^{3k} \equiv b^{3k} \pmod p$. So, combining these gives $a \equiv b \pmod p$.

Then,

$$a^2 + ab + b^2 \equiv 0 \pmod p \implies 3a^2 \equiv 0 \pmod p \implies a^2 \equiv 0 \pmod p,$$

where the last step can be done since $\gcd(3, p) = 0$. Finally, since p is a prime, the last equivalence implies $a \equiv 0 \pmod p$, as desired. □

Solution 4

First, note that the right hand side has the same sign as $q - p$ since $2q - 1$ is odd. Clearly the left hand side is positive, so this implies $q - p$ is positive as well, i.e. $q > p$.

Now, let $(p+q)^p = (q-p)^{2q-1} = N$. We claim that N is a power of two: indeed, if $r \mid N$ for some prime r, then since r is a prime, we get that $r \mid p+q$ and $r \mid q-p$. But then, r must divide both the sum and difference of these two expressions, which are $2q$ and $2p$, respectively. So $r \mid 2q$ and $r \mid 2p$. However, p, q are distinct primes, so it is not possible for r to be a multiple of p or q and still satisfy those conditions. That is, we have that $r \mid 2$. Thus 2 is the only prime that can divide N. So N is a power of two, as claimed.

This means that $p+q = 2^u$ and $q-p = 2^v$, for some integers u, v. Also clearly, with this definition, $2^v < 2^u$. Then note that

$$\begin{aligned} 2^u - 2^v &= (p+q) - (q-p) \\ \implies 2^v(2^{u-v} - 1) &= 2p. \end{aligned}$$

If $p = 2$, then $2^u, 2^v$ differ by 4; but the only such powers of 2 are 4 and 8, and $q - p = 4$, $p + q = 8$ does not give a prime value of q. Hence it is not possible for $p = 2$ to hold.

Thus, p is an odd prime, so in the equation $2^v(2^{u-v} - 1) = 2p$, the power of 2 on the left side is 2^v and on the right side is just 2. Hence $2^v = 2$. That is, $q - p = 2 \implies q = p + 2$.

Plugging this back into the original equation makes it

$$(2p+2)^p = 2^{2p+3}.$$

Now, note that if $p > 4$, then

$$(2p+2)^p > (2p)^p > (2 \cdot 4)^p > 8^p = 2^{2p+p} > 2^{2p+3}.$$

Hence if $p > 4$ it is not possible for the equation to hold. Since p also must be an odd prime, this means the only possible value of p is 3. Checking this, we see that it gives the solution of $\boxed{(p, q) = (3, 5)}$, which is the only solution. □

Solution 5

Note that we can factor the expression as

$$\begin{aligned} n^8 + n + 1 &= n^8 - n^2 + n^2 + n + 1 \\ &= n^2(n^6 - 1) + (n^2 + n + 1) \\ &= n^2(n^3 - 1)(n^3 + 1) + (n^2 + n + 1) \\ &= n^2(n-1)(n^2 + n + 1)(n^3 + 1) + (n^2 + n + 1) \\ &= (n^2 + n + 1)(n^2(n-1)(n^3 + 1) + 1). \end{aligned}$$

Both of these factors are evidently positive integers, so one of them must be 1. If $n^2 + n + 1 = 1$, then the only positive integer solution is $n = 1$. If $n^2(n-1)(n^3+1)+1 = 1$, then once again, the only positive integer solution is $n = 1$. Thus, the only such integer is $\boxed{n = 1}$. □

Solution 6

The first condition gives $y \equiv 3x \pmod 5$, and substituting this into the second gives

$$3x^2 \equiv 2 \pmod 5 \implies x^2 \equiv 4 \pmod 5.$$

This means x is either 2 or 3 modulo 5. For each choice of x, there is a unique choice of y modulo 5, meaning that there are 4 possibilities for y out of $\{1, \ldots, 20\}$. Since there are 8 possibilities for x in this range (i.e. values that are 2 or 3 modulo 5), the answer is then $8 \cdot 4 = \boxed{32}$. □

Solution 7

Using Euler's Theorem, since $\phi(49) = 42$, we have that

$$\begin{aligned} a_83 &\equiv 6^{83} + 8^{83} \\ &\equiv 6^{84} \cdot \frac{1}{6} + 8^{84} \cdot \frac{1}{8} \\ &\equiv \frac{1}{6} + \frac{1}{8} \\ &\equiv \frac{14}{48} \\ &\equiv \frac{14}{-1} \\ &\equiv -14 \\ &\equiv \boxed{35} \pmod{49}. \end{aligned}$$

Solution 8

Following the hint, we have that N is even and that $N^3 \equiv 888 \pmod{125}$. First, note that this means $N^3 \equiv 3 \pmod 5$, which implies that $N \equiv 2 \pmod 5$. So, $N = 5k + 2$ for some k.

Then, we have $N^3 \equiv 13 \pmod{25}$, so

$$13 \equiv (5k+2)^3 \equiv 125k + 25k(2)(3) + 5k(4)(3) + 8 \equiv 10k + 8 \pmod{25}.$$

So $5 \equiv 10k \pmod{25}$, which implies that $k = 3 \pmod 5$. That is, $k = 5j + 3$ for some j, so $N = 5k + 2 = 25j + 17$. Finally, note that

$$(25j + 17)^3 = (25j)^3 + (25j)^2(17)(3) + 25j(289)(3) + 17^3,$$

and taking this expression modulo 125 and setting it to $888 \equiv 13 \pmod{125}$ results in

$$25j(289)(3) + 17^3 \equiv 13 \pmod{125} \implies 25j(49)(3) \equiv 100 \pmod{125}.$$

This equivalence implies $j(49)(3) \equiv 4 \pmod 5$, or that $j \equiv 2 \pmod 5$. So, we have $N \equiv 25j + 17 \equiv 67 \pmod{125}$.

Also, we know N must be even, and the smallest such number is $125 + 67 = \boxed{192}$. □

Solution 9

Similar to above. Let the number be N. Analyzing this modulo 8, we have that $N^3 \equiv 7 \pmod 8$, meaning that $N \equiv 7 \pmod 8$.

Now we find N modulo 125. First note that $N^3 \equiv 2 \pmod 5$, which implies $N \equiv 3 \pmod 5$. Hence the value of N modulo 125 can be written as $5k + 3$ for some k. Then, note that since $N^3 \equiv 7 \pmod{25}$,

$$7 \equiv N^3 \equiv (5k+3)^3 \equiv 125k^3 + 225k^2 + 135k + 27 \equiv 10k + 2 \pmod{25}.$$

This implies $5 \equiv 10k \pmod{25}$, so $k \equiv 3 \pmod 5$. That is, $k = 5j + 3$ for some j, so $N = 5k + 3 = 25j + 18$. Finally, note that

$$(25j + 18)^3 = (25j)^3 + (25j)^2(17)(3) + 25j(324)(3) + 18^3.$$

Taking this expression modulo 125 and setting it equal to $207 \equiv 82 \pmod{125}$ gives

$$25j(324)(3) + 18^3 \equiv 82 \pmod{125} \implies 25j(324)(3) \equiv 0 \pmod{125}.$$

This implies $j \equiv 0 \pmod 5$. Hence $N \equiv 25j + 18 \equiv 18 \pmod{125}$.

Also, $N \equiv 7 \pmod 8$, and the smallest N satisfying both of these is $N = \boxed{143}$. □

Solution 10

The answer is 111. Consider the seven numbers $26, 39, 52, \ldots, 104$; each of these numbers is one of the possible residues modulo 7, so for each of these residue values, adding a multiple of 7 that is at least 14 to the corresponding multiple of 13 listed above will give a good representation of some number. Thus the largest numbers which cannot be represented in the desired form will be those numbers which are $26, 39, \ldots, 104$ plus just 7, the largest of which is $\boxed{111}$. □

Solution 11

First, it is easy to verify that $n = 1, 2$ do not work; now consider $n \geq 3$. By considering the expression modulo 8 and 125, we see that the condition is equivalent to

$$n \equiv 5^n \pmod 8, \qquad n \equiv 2^n \pmod{125}.$$

For the first condition, since 5^n is equivalent to 1 for even n and 5 for odd n, we must have $n \equiv 5 \pmod 8$ in order for this condition to hold.

To handle the second condition, first we look at the condition $n \equiv 2^n \pmod 5$. We know $n \equiv 5 \pmod 8$; in particular, $n \equiv 1 \pmod 4$, and since $2^4 \equiv 1 \pmod 5$, this means that $2^n \equiv 2^1 \pmod 5$. Hence, $n \equiv 2^n \equiv 2 \pmod 5$.

Then, combining this with $n \equiv 1 \pmod 4$ gives $n \equiv 17 \pmod{20}$. Now, since $2^{20} \equiv 1 \pmod{25}$ by Euler's Thm and we know $n \equiv 2^n \pmod{25}$, we have

$$n \equiv 2^n \equiv 2^{17} \pmod{25}.$$

We compute this as $2^{17} \equiv 2^{20} \cdot (2^{-1})^3 \equiv 13^3 \equiv 22 \pmod{25}$.

Then, combining this with $n \equiv 1 \pmod 4$ gives $n \equiv 97 \pmod{100}$. Since $2^{100} \equiv 1 \pmod{125}$ by Euler's Thm and we know $n \equiv 2^n \pmod{125}$, we have

$$n \equiv 2^n \equiv 2^{97} \pmod{100}.$$

Now we compute this as

$$2^{97} \equiv 2^{100} \cdot (2^{-1})^3 \equiv 63^3 \equiv 47 \pmod{125}.$$

Then, combining this with $n \equiv 5 \pmod 8$ gives $n \equiv 797 \pmod{1000}$, so the answer is $\boxed{797}$. □