Solution 1

We claim that $3p \mid m - 1$.

First, note that $8^p - 1 = (8 - 1)(8^{p-1} + 8^{p-2} + \ldots + 1)$. Hence $m = 8^{p-1} + 8^{p-2} + \ldots + 1$. This expression contains an odd number of terms since $p > 7$ is prime and therefore odd, and there are $p$ terms in the sum. Furthermore, they alternate between 1 and $-1$ modulo 3 based on the parity of the exponent, so we see that $m \equiv 1 \pmod{3}$. Hence $3 \mid m - 1$.

Also, by Fermat’s Little Theorem, and the fact that gcd(7, $p$) = 1,

$$m \equiv \frac{8^p - 1}{7} \equiv \frac{8 - 1}{7} \equiv 1 \pmod{p}.$$ 

Hence $p \mid m - 1$. Finally, since gcd(3, $p$) = 1, combining these two facts gives $3p \mid m - 1$. Thus, $m - 1 = 3pk$ for some positive integer $k$, so $2^{m-1} \equiv 2^{3pk} \equiv (8^p)^k \pmod{m}$.

Furthermore, by the definition of $m$,

$$\frac{8^p - 1}{7} \equiv m \equiv 0 \pmod{m} \implies 8^p \equiv 1 \pmod{m}$$

from manipulating the equivalence. Hence $(8^p)^k \equiv 1^k \equiv 1 \pmod{m}$, as desired. □

Solution 2

Due to the definition of remainder, the given conditions imply that $b < a$ and $a - 2 < b$. So, $a - 2 < b < a$, but the only number between $a - 2$ and $a$ is $a - 1$, so this implies $b = a - 1$. Then, this means $n \equiv b \equiv -1 \pmod{a}$, and $n \equiv a - 2 \equiv -1 \pmod{b}$. Also, since gcd($a$, $b$) = 1 (since they are consecutive integers), we can combine these to conclude $n \equiv -1 \pmod{ab}$. That is, $n \equiv -1 \pmod{a(a - 1)}$, where $a \geq 6$ (since $a - 1 = b \geq 5$).

Now we simply count the answer. Since $n \geq 100$: for $a = 6$, $n = 35, 71$ work; for $a = 7$, $n = 41, 83$ work; for $n = 8$, $n = 55$ works; for $a = 9$, $n = 71$ works; for $a = 10$, $n = 89$ works. So there are 7 solutions. □

Solution 3

Note that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, so the given condition implies $p \mid a^3 - b^3$. That is, $a^3 \equiv b^3 \pmod{p}$. Also, by Fermat’s Little Theorem, $a^{3k+1} \equiv 1 \equiv b^{3k+1} \pmod{p}$. From the first derived fact, we also know $a^{3k} \equiv b^{3k} \pmod{p}$. So, combining these gives $a \equiv b \pmod{p}$.

Then,

$$a^2 + ab + b^2 \equiv 0 \pmod{p} \implies 3a^2 \equiv 0 \pmod{p} \implies a^2 \equiv 0 \pmod{p},$$

where the last step can be done since gcd(3, $p$) = 0. Finally, since $p$ is a prime, the last equivalence implies $a \equiv 0 \pmod{p}$, as desired. □

Solution 4

First, note that the right hand side has the same sign as $q-p$ since $2q-1$ is odd. Clearly the left hand side is positive, so this implies $q-p$ is positive as well, i.e. $q > p$. 
Now, let \((p + q)^p = (q - p)^{2q-1} = N\). We claim that \(N\) is a power of two: indeed, if \(r | N\) for some prime \(r\), then since \(r\) is a prime, we get that \(r | p + q\) and \(r | q - p\). But then, \(r\) must divide both the sum and difference of these two expressions, which are \(2q\) and \(2p\), respectively. So \(r | 2q\) and \(r | 2p\). However, \(p, q\) are distinct primes, so it is not possible for \(r\) to be a multiple of \(p\) or \(q\) and still satisfy those conditions. That is, we have that \(r | 2\). Thus \(2\) is the only prime that can divide \(N\). So \(N\) is a power of two, as claimed.

This means that \(p + q = 2^u\) and \(q - p = 2^v\), for some integers \(u, v\). Also clearly, with this definition, \(2^v < 2^u\). Then note that
\[
2^u - 2^v = (p + q) - (q - p) \\
\Rightarrow 2^v(2^{u-v} - 1) = 2p.
\]
If \(p = 2\), then \(2^u, 2^v\) differ by 4; but the only such powers of 2 are 4 and 8, and \(q - p = 4, p + q = 8\) does not give a prime value of \(q\). Hence it is not possible for \(p = 2\) to hold.

Thus, \(p\) is an odd prime, so in the equation \(2^v(2^{u-v} - 1) = 2p\), the power of 2 on the left side is \(2^v\) and on the right side is just 2. Hence \(2^v = 2\). That is, \(q - p = 2 \Rightarrow q = p + 2\).

Plugging this back into the original equation makes it
\[
(2p + 2)^p = 2^{2p+3}.
\]
Now, note that if \(p > 4\), then
\[
(2p + 2)^p > (2p)^p > (2 \cdot 4)^p > 8^p = 2^{2p+p} > 2^{2p+3}.
\]
Hence if \(p > 4\) it is not possible for the equation to hold. Since \(p\) also must be an odd prime, this means the only possible value of \(p\) is 3. Checking this, we see that it gives the solution of \((p, q) = (3, 5)\), which is the only solution.

\[\square\]

**Solution 5**

Note that we can factor the expression as
\[
n^8 + n + 1 = n^8 - n^2 + n^2 + n + 1 \\
= n^2(n^6 - 1) + (n^2 + n + 1) \\
= n^2(n^3 - 1)(n^3 + 1) + (n^2 + n + 1) \\
= n^2(n - 1)(n^2 + n + 1)(n^3 + 1) + (n^2 + n + 1) \\
= (n^2 + n + 1)(n^2(n - 1)(n^3 + 1) + 1).
\]
Both of these factors are evidently positive integers, so one of them must be 1. If \(n^2 + n + 1 = 1\), then the only positive integer solution is \(n = 1\). If \(n^2(n - 1)(n^3 + 1) + 1 = 1\), then once again, the only positive integer solution is \(n = 1\). Thus, the only such integer is \(n = 1\). \[\square\]

**Solution 6**

The first condition gives \(y \equiv 3x \pmod{5}\), and substituting this into the second gives
\[
3x^2 \equiv 2 \pmod{5} \implies x^2 \equiv 4 \pmod{5}.
\]
This means \(x\) is either 2 or 3 modulo 5. For each choice of \(x\), there is a unique choice of \(y\) modulo 5, meaning that there are 4 possibilities for \(y\) out of \(\{1, \ldots, 20\}\). Since there are 8 possibilities for \(x\) in this range (i.e. values that are 2 or 3 modulo 5), the answer is then \(8 \cdot 4 = 32\). \[\square\]
Solution 7

Using Euler’s Theorem, since $\phi(49) = 42$, we have that

\[ a^{83} \equiv 6^{83} + 8^{83} \]
\[ \equiv 6^{84} \cdot \frac{1}{6} + 8^{84} \cdot \frac{1}{8} \]
\[ \equiv \frac{1}{6} + \frac{1}{8} \]
\[ \equiv \frac{14}{48} \]
\[ \equiv \frac{14}{18} \]
\[ \equiv -1 \]
\[ \equiv -14 \]
\[ \equiv 35 \pmod{49}. \]

Solution 8

Following the hint, we have that $N$ is even and that $N^3 \equiv 888 \pmod{125}$. First, note that this means $N^3 \equiv 3 \pmod{5}$, which implies that $N \equiv 2 \pmod{5}$. So, $N = 5k + 2$ for some $k$.

Then, we have $N^3 \equiv 13 \pmod{25}$, so

\[ 13 \equiv (5k + 2)^3 \equiv 125k + 25k(2)(3) + 5k(4)(3) + 8 \equiv 10k + 8 \pmod{25}. \]

So $5 \equiv 10k \pmod{25}$, which implies that $k \equiv 3 \pmod{5}$. That is, $k = 5j + 3$ for some $j$, so $N = 5k + 2 = 25j + 17$. Finally, note that

\[ (25j + 17)^3 = (25j)^3 + (25j)^2(17)(3) + 25j(289)(3) + 17^3, \]

and taking this expression modulo 125 and setting it to 888 $\equiv 13 \pmod{125}$ results in

\[ 25j(289)(3) + 17^3 \equiv 13 \pmod{125} \Rightarrow 25j(49)(3) \equiv 100 \pmod{125}. \]

This equivalence implies $j(49)(3) \equiv 4 \pmod{5}$, or that $j \equiv 2 \pmod{5}$. So, we have $N \equiv 25j + 17 \equiv 67 \pmod{125}$.

Also, we know $N$ must be even, and the smallest such number is $125 + 67 = 192$. \hfill $\Box$

Solution 9

Similar to above. Let the number be $N$. Analyzing this modulo 8, we have that $N^3 \equiv 7 \pmod{8}$, meaning that $N \equiv 7 \pmod{8}$.

Now we find $N$ modulo 125. First note that $N^3 \equiv 2 \pmod{5}$, which implies $N \equiv 3 \pmod{5}$. Hence the value of $N$ modulo 125 can be written as $5k + 3$ for some $k$. Then, note that since $N^3 \equiv 7 \pmod{25}$,

\[ 7 \equiv N^3 \equiv (5k + 3)^3 \equiv 125k^3 + 225k^2 + 135k + 27 \equiv 10k + 2 \pmod{25}. \]

This implies $5 \equiv 10k \pmod{25}$, so $k \equiv 3 \pmod{5}$. That is, $k = 5j + 3$ for some $j$, so $N = 5k + 3 = 25j + 18$.

Finally, note that

\[ (25j + 18)^3 = (25j)^3 + (25j)^2(17)(3) + 25j(324)(3) + 18^3. \]
Taking this expression modulo 125 and setting it equal to 207 \equiv 82 \pmod{125} gives
\[25j(324)(3) + 18^3 \equiv 82 \pmod{125} \implies 25j(324)(3) \equiv 0 \pmod{125}.
\]
This implies \(j \equiv 0 \pmod{5}\). Hence \(N \equiv 25j + 18 \equiv 18 \pmod{125}\).

Also, \(N \equiv 7 \pmod{8}\), and the smallest \(N\) satisfying both of these is \(N = 143\). \(\square\)

**Solution 10**

The answer is 111. Consider the seven numbers 26, 39, 52, \ldots, 104; each of these numbers is one of the possible residues modulo 7, so for each of these residue values, adding a multiple of 7 that is at least 14 to the corresponding multiple of 13 listed above will give a good representation of some number. Thus the largest numbers which cannot be represented in the desired form will be those numbers which are 26, 39, \ldots, 104 plus just 7, the largest of which is 111. \(\square\)

**Solution 11**

First, it is easy to verify that \(n = 1, 2\) do not work; now consider \(n \geq 3\). By considering the expression modulo 8 and 125, we see that the condition is equivalent to
\[n \equiv 5^n \pmod{8}, \quad n \equiv 2^n \pmod{125}.
\]

For the first condition, since \(5^n\) is equivalent to 1 for even \(n\) and 5 for odd \(n\), we must have \(n \equiv 5 \pmod{8}\) in order for this condition to hold.

To handle the second condition, first we look at the condition \(n \equiv 2^n \pmod{5}\). We know \(n \equiv 5 \pmod{8}\); in particular, \(n \equiv 1 \pmod{4}\), and since \(2^4 \equiv 1 \pmod{5}\), this means that \(2^n \equiv 2^1 \pmod{5}\). Hence, \(n \equiv 2^n \equiv 2 \pmod{5}\).

Then, combining this with \(n \equiv 1 \pmod{4}\) gives \(n \equiv 17 \pmod{20}\). Now, since \(2^{20} \equiv 1 \pmod{25}\) by Euler’s Thm and we know \(n \equiv 2^n \pmod{25}\), we have
\[n \equiv 2^n \equiv 2^{17} \pmod{25}.
\]
We compute this as \(2^{17} = 2^{20} \cdot (2^{-1})^3 = 13^3 = 22 \pmod{25}\).

Then, combining this with \(n \equiv 1 \pmod{4}\) gives \(n \equiv 97 \pmod{100}\). Since \(2^{100} \equiv 1 \pmod{125}\) by Euler’s Thm and we know \(n \equiv 2^n \pmod{125}\), we have
\[n \equiv 2^n \equiv 2^{97} \pmod{100}.
\]
Now we compute this as
\[2^{97} = 2^{100} \cdot (2^{-1})^3 = 63^3 = 47 \pmod{125}.
\]
Then, combining this with \(n \equiv 5 \pmod{8}\) gives \(n \equiv 797 \pmod{1000}\), so the answer is 797. \(\square\)