ORMC Olympiad Group Winter 2022
Week 5 Solutions

Sumith Nalabolu

January 21, 2022

Solution 1

By Fermat's Little Theorem, $2^6 \equiv 1 \pmod 7$. Hence it suffices to compute the exponent modulo 6. Note that $1000^{1000} \equiv 4^{1000} \pmod 6$. But note that any power of 4 is just $4 \pmod 6$, so $1000^{1000} \equiv 4^{1000} \equiv 4 \pmod 6$. This means that

$$2^{1000^{1000}} \equiv 2^4 \pmod 7,$$

by Fermat's Little Theorem. Then clearly $2^4 \equiv \boxed{2} \pmod 7$. $\square$

Solution 2

We compute it separately modulo 3 and 5. Clearly 1000^{1000} is even, so

$$2^{1000^{1000}} \equiv (-1)^{1000^{1000}} \equiv 1 \pmod 3,$$

and similarly, $n = 500 \cdot 1000^{999}$ is even, so

$$2^{1000^{1000}} \equiv 2^{2 \cdot 500 \cdot 1000^{999}} \equiv (2^2)^n \equiv -1^n \equiv 1 \pmod 5.$$

Hence we can conclude that $2^{1000^{1000}} \equiv \boxed{1} \pmod{15}$. $\square$

Solution 3

The problem is asking us to compute this number modulo 1000. Since $2013 \equiv 13 \pmod{1000}$, we have $2013^{1000} \equiv 13^{1000} \pmod{1000}$. Now, to compute this value, we consider it modulo 8 and 125 separately (since $1000 = 2^3 \cdot 5^3$).

First, note that $13^{1000} \equiv 5^{1000} \equiv 1 \pmod 8$, where the last step is due to the fact that $5^2 \equiv 1 \pmod 8$.

Now, note that by Euler's Theorem and the result of Problem 5 above, $13^{100} \equiv 1 \pmod{125}$. Thus $13^{1000} \equiv 1^{10} \equiv 1 \pmod{125}$. Combining this with the above gives that the number is in fact $\boxed{1} \pmod{1000}$. $\square$

Solution 4

In order to compute this modulo 1000, we compute it modulo 8 and 125. Also, clearly it is equivalent to compute the last three digits of 17^{2017}. First, note that

$$17^{2017} \equiv 1^{2017} \equiv 1 \pmod 8.$$

Also, by Euler's Theorem and the result of Problem 5, we have $17^{100} \equiv 1 \pmod{125}$. Hence $17^{2017} \equiv 17^{17} \pmod{125}$. Now, it is possible to just compute this as

$$17^{17} \equiv 17 \cdot (39)^8 \equiv 17 \cdot (21)^4 \equiv 17(66)^2 \equiv 52 \pmod{125}.$$

Then, combining this with the fact that it is $1 \pmod 8$ gives that the number is $\boxed{177} \pmod{1000}$. $\square$

Solution 5

We must calculate the expression modulo 1000. Clearly, it is equivalent to $3^{2002^{2001}} \pmod{1000}$. Now, note that $2002^{2001} = 2 \cdot (1001 \cdot 2002^{2000})$; for simplicity, let $n = 1001 \cdot 2002^{2000}$; then, we have

$$3^{2002^{2001}} \equiv 3^{2 \cdot n} \equiv (3^2)^n \equiv 9^n \pmod{1000}.$$

Now, by the binomial theorem,

$$9^n = (10-1)^n = \sum_{i=0}^{n} 10^i (-1)^{n-i} \binom{n}{i}.$$

Clearly any term corresponding to $i \geq 3$ will simply be 0 modulo 1000. Hence it suffices to sum the first three terms:

$$9^n \equiv (-1)^n \binom{n}{0} + 10(-1)^{n-1}\binom{n}{1} + 100(-1)^{n-2}\binom{n}{2} \pmod{1000}.$$

Furthermore, note that n is even, so $(-1)^n = 1$; also, we will compute n modulo 1000: note that $n \equiv 1 \cdot 2^{2000} \pmod{1000}$, and to compute this, we separately compute it modulo 8 and 125 (since $1000 = 2^3 \cdot 5^3$).

Clearly, $n \equiv 2^{2000} \equiv 0 \pmod 8$. Furthermore, by Euler's Theorem and the property shown in problem 5, $2^{100} \equiv 1 \pmod{125}$, implying that $2^{2000} \equiv 1 \pmod{125}$. Combining these gives that $n \equiv 376 \pmod{1000}$.

We must compute

$$9^n \equiv (-1)^n \binom{n}{0} + 10(-1)^{n-1}\binom{n}{1} + 100(-1)^{n-2}\binom{n}{2} \equiv 1 - 10n + 100n(n-1)/2 \pmod{1000},$$

and we know $n \equiv 376 \pmod{1000}$, so now computation gives $9^n \equiv \boxed{241} \pmod{1000}$. □

Solution 6

First we consider the equation modulo p. By Fermat's little theorem, $3^p \equiv 3 \pmod 3$, so the equation becomes

$$3 - 4 \equiv m^2 \pmod p \implies -1 \equiv m^2 \pmod p.$$

Then, note that $(-1)^{\frac{p-1}{2}} \equiv (m^2)^{\frac{p-1}{2}} \pmod p$. But the right side simplifies to $m^{p-1} \equiv 1 \pmod p$. Hence we have $(-1)^{\frac{p-1}{2}} \equiv 1 \pmod p$. This means $\frac{p-1}{2}$ must be even, i.e. $p \equiv 1 \pmod 4$.

Now, considering the equation modulo 4 gives

$$7(1) + (-1)^p - 4 \equiv m^2 \pmod p \implies 7 - 1 - 4 \equiv 2 \equiv m^2 \pmod 4.$$

Note that $(-1)^p \equiv -1 \pmod 4$ because p is odd. So $m^2 \equiv 2 \pmod 4$. But this does not hold for any integer m, so there are in fact no solutions to the equation. □

Solution 7

First note that if $5 \mid x$, then $x^2 \equiv 0 \pmod{25}$. Now we consider all other values of x modulo 25. To count the number of distinct values of x^2, we will see in what cases $x^2 \equiv y^2 \pmod{25}$ holds when $x \neq y$. From factoring this with difference of squares, this implies $25 \mid (x-y)(x+y)$. Note that since $5 \nmid y$, and since the two terms $x-y$ and $x+y$ differ by $2y$, they cannot both be multiples of 5. Hence one of them must be divisible by 25. But $x \neq y$, so we must have $25 \mid x+y$. That is, $x + y = 25$, as we are picking x, y from $\{1, \ldots, 24\}$.

The number of $x \in \{0, \ldots, 24\}$ which are not multiples of 5 is 20, so this contributes 10 distinct perfect squares mod 25. Then, 0 is also a perfect square as noted in the beginning, so there are $\boxed{11}$ perfect squares mod 25 in total. □

Solution 8

The answer is odd n. To show this, first consider any odd n; we must construct a complete residue class satisfying the desired condition. Let the set of a_i with $a_i = i$ for $0 \leq i \leq n-1$ be the complete residue class. Then, the set of $a_i + i$ contains all distinct numbers: if $a_i + i \equiv a_j + j \pmod{n}$, then since each $a_i = i$, this means $2i \equiv 2j \pmod{n}$. But then since n is odd, this simplifies to $i \equiv j \pmod{n}$ (note that this step does not hold when n is even). Since $i, j \in \{0, \ldots, n-1\}$ this implies $i = j$, so we see that the set $\{a_0, a_1 + 1, \ldots, a_{n-1} + n - 1\}$ in fact consists of n distinct numbers, implying that it is a complete residue class.

Also, we must show that even n do not satisfy the condition. Assume for the sake of contradiction that there does some exist some $\{a_i\}$ such that $\{a_i + i\}$ is also a complete residue classes. Then they contain the same elements in some (possibly different) permutation; in particular, this means that the sums of the elements for each set must be equal. That is,

$$\begin{aligned}
\sum_{i=0}^{n-1} a_i &\equiv \sum_{i=0}^{n-1} (a_i + i) \pmod{n} \\
\implies 0 &\equiv \sum_{i=0}^{n-1} i \pmod{n} \\
\implies 0 &\equiv \frac{(n-1)n}{2} \pmod{n}.
\end{aligned}$$

But note that for even n, $n-1$ is odd so then it is clear that $n \nmid (n-1)\frac{n}{2}$ by considering the power of 2 in each expression. Hence even n cannot satisfy the desired condition. $\square$

Solution 9

The answer is odd n. The solution is analogous to the above. For any odd n, to construct a residue class satisfying the condition, simply take the set of a_i with $a_i = i$ for each $0 \leq i \leq n-1$. Then, the set of a_i contains all distinct numbers due to the following: the i-th number in this set is $i + 3i = 4i$, so if $4i \equiv 4j \pmod{n}$, then since n is odd we can cancel the factor of 4 to obtain $i \equiv j \pmod{n}$. This cannot happen for distinct indices $0 \leq i, j \leq n-1$. So this implies all of the $a_i + i$ are distinct, meaning that $a_0, a_1 + 3, \ldots, a_{n-1} + 3(n-1)$ is a complete residue class.

So, all odd n satisfy the condition. Now we show that even n do not work. Suppose that n satisfies the given condition. Then, there exists a complete residue class $\{a_i : 0 \leq i \leq n-1\}$ such that $\{a_i + 3i : 0 \leq i \leq n-1\}$ is also a complete residue class, so these two sets contain the same elements, in some (possibly different) orders. So, they have the same sum:

$$\begin{aligned}
\sum_{i=0}^{n-1} a_i &\equiv \sum_{i=0}^{n-1} (a_i + 3i) \pmod{n} \\
\implies 0 &\equiv \sum_{i=0}^{n-1} 3i \pmod{n} \\
\implies 0 &\equiv \frac{3(n-1)n}{2} \pmod{n}.
\end{aligned}$$

But for even n, $n-1$ is odd, so it is clear that $n \nmid 3(n-1)\frac{n}{2}$ by considering the power of 2 in each expression. Hence even n cannot satisfy the desired condition. $\square$

Solution 10

No. Suppose $\{a_1, 2a_2, \ldots, (p-1)a_{p-1}\}$ is a complete residue class where $\{a_1, \ldots, a_{p-1}\}$ also is one. Then, since each of these sets are just some permutation of $\{1, \ldots, p-1\}$, we have that

$$\prod_{i=1}^{p-1} ia_i \equiv (p-1)! \equiv -1 \pmod{p},$$

where the last step is by Wilson's theorem; but also, note that

$$\prod_{i=1}^{p-1} ia_i \equiv \left(\prod_{i=1}^{p-1} i\right)\left(\prod_{i=1}^{p-1} a_i\right) \equiv (p-1)!(p-1)! \equiv 1 \pmod{p},$$

again by Wilson's theorem. So this implies that $-1 \equiv 1 \pmod{p}$, but this is not possible since $p > 2$. $\square$

Solution 11

Due to the definition of remainder, the given conditions imply that $b < a$ and $a - 2 < b$. So, $a - 2 < b < a$, but the only number between $a - 2$ and a is $a - 1$, so this implies $b = a - 1$. Then, this means $n \equiv b \equiv -1 \pmod{a}$, and $n \equiv a - 2 \equiv -1 \pmod{b}$. Also, since $\gcd(a, b) = 1$ (since they are consecutive integers), we can combine these to conclude $n \equiv -1 \pmod{ab}$. That is, $n \equiv -1 \pmod{a(a-1)}$, where $a \geq 6$ (since $a - 1 = b \geq 5$).

Now we simply count the answer. Since $n \geq 100$: for $a = 6$, $n = 35, 71$ work; for $a = 7$, $n = 41, 83$ work; for $n = 8$, $n = 55$ works; for $a = 9$, $n = 71$ works; for $a = 10$, $n = 89$ works. So there are $\boxed{7}$ solutions. $\square$

Solution 12

(a) No. By the divisibility rules, it is easy to verify that $3 \mid A$ but $9 \nmid A$. Hence it is not possible for A to be a perfect square, as the exponent of 3 in its prime factorization is 1. $\square$

(b) No. Suppose A is a square; then, consider the rightmost six in A, which must exist since there are 600 sixes in A. Suppose this is followed by k zeroes to its right. We claim that k must be even.

Indeed, note that either the largest power of 2 dividing A is k, or the largest power of 5 dividing A is k (or both of these are true), because if not, then $2^{k+1} \cdot 5^{k+1} \mid A$, which would imply that A ends in at least $k + 1$ zeroes. But then, if k is odd, then either 2 or 5 (or both) has an odd exponent (i.e. k) in the prime factorization of A, so A is not a perfect square. Hence k must be even.

This means $A = A_1 \cdot 10^{2k_1}$ where $k_1 = k/2 \in \mathbb{Z}_{\geq 0}$. In particular, since A and 10^{2k_1} are squares, A_1 is a square number as well, and we also know that it ends in six and contains only six and zero. Thus, the last two digits must be either 06 or 66. But in either case, this implies $A_1 \equiv 2 \pmod{4}$, contradicting the fact that A_1 is a square number. Hence, it is not possible for A to be a square. $\square$

Solution 13

Notice that $1000 \equiv 1 \pmod{27}$; hence, for each $k = 1, 2, 3$, by splitting the number M_{400} into groups of three digits and summing those three digit numbers, we have that

$$M_{400} \equiv \sum_{i=1}^{400} i - \sum_{i=1}^{100} i \equiv \frac{400(401) - 100(101)}{2} \pmod{3^k}.$$

The expression simplifies to 50(1503). We see that this expression is divisible by 9 but not by 27, implying that M_{400} is divisible by 9 but not 27, so the answer is $\boxed{2}$. $\square$

Solution 14

Rearranging the equation gives

$$p^2 - 1 = 2q^2 \implies (p-1)(p+1) = 2q^2.$$

Note that the right hand side is even, so the left side is as well. Furthermore, $p-1$ and $p+1$ have the same parity; since their product is even, they must both be even, so in fact $4 \mid (p-1)(p+1)$. So $4 \mid 2q^2 \implies 2 \mid q^2$, and since q is prime, this implies $q = 2$. Then solving the equation gives $p = 3$, so the only solution is $\boxed{(3,2)}$. $\square$

Solution 15

We prove the contrapositive. Suppose n is not prime. Then, there exists a positive integer a with $1 < a < n$ and $\frac{n}{a} = b \in \mathbb{N}$. (i.e. some pair of divisors). Then, we can factor

$$2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1^b = (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \ldots + 1).$$

Clearly $1 < 2^a - 1 < n$, so this implies $2^n - 1$ has a divisor other than 1 and itself, and is therefore not prime. Hence if $2^n - 1$ is prime, then n is prime. $\square$

Solution 16

We prove the contrapositive. Suppose $p \mid n$ for some odd prime p, and with $pq = n$.

$$2^n = (2^p)^q + 1^q = (2^p + 1)(2^{p-1} - 2^{p-2} + \ldots + 1).$$

Clearly $1 < 2^p + 1 < 2^n + 1$, so this implies $2^n + 1$ is not a prime number. Hence if $2^n + 1$ is prime, then no odd prime divides n; that is, n must be a power of two. $\square$

Solution 17

No; it can be verified that F_5 is not a prime number, and that $641 \mid F_5$. $\square$

Solution 18

In one case, note that if x is odd, then the left hand side is even. Since z is prime, this implies $z = 2$, but then there are no solutions in primes to $x^y + 1 = 2$. Hence there are no solutions in this case.

In the other case, if x is even, i.e. $x = 2$, then $2^y + 1 = z$ where z is prime. Then, by problem 16, this implies that y is a power of 2. However, y must also be prime, so this means $y = 2$. Thus, $z = 5$. The answer is then only the triple $\boxed{(2,2,5)}$. $\square$

Solution 19

Note that $a^3 - b^3 = (a-b)(a^2 + ab + b^2$, so the given condition implies $p \mid a^3 - b^3$. That is, $a^3 \equiv b^3 \pmod{p}$. Also, by Fermat's Little Theorem, $a^{3k+1} \equiv 1 \equiv b^{3k+1} \pmod{p}$. From the first derived fact, we also know $a^{3k} \equiv b^{3k} \pmod{p}$. So, combining these gives $a \equiv b \pmod{p}$.

Then,

$$a^2 + ab + b^2 \equiv 0 \pmod{p} \implies 3a^2 \equiv 0 \pmod{p} \implies a^2 \equiv 0 \pmod{p},$$

where the last step can be done since $\gcd(3, p) = 0$. Finally, since p is a prime, the last equivalence implies $a \equiv 0 \pmod{p}$, as desired. $\square$