ORMC Olympiad Group Winter 2022
Week 4 Solutions

Sumith Nalabolu

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Solution 1

First note that the units digit of 7^k repeats in a cycle of length 4. That is, for positive integers $k \equiv j \pmod 4$, then $7^k \equiv 7^j \pmod{10}$. So, it suffices to compute the exponent modulo 4.

Note that $7^7 \equiv (-1)^7 \equiv -1 \equiv 3 \pmod 4$. Hence, $7^{(7^7)} \equiv 7^3 \pmod{10}$, and this can be computed to have units digit of $\boxed{3}$.

Solution 2

By Fermat's Little Theorem, $2^{12} \equiv 1 \pmod{13}$. Hence if $x \equiv y \pmod{12}$, then $2^x \equiv 2^y \pmod{13}$. Thus, it suffices to simplify the exponent $50^{50} \pmod{12}$.

To compute this, we consider it modulo 3 and 4 separately. First, note that clearly $50^{50} \equiv 0 \pmod 4$. Furthermore, $50^{50} \equiv (-1)^{50} \equiv 1 \pmod 3$. Then, we can see that the only residue modulo 12 which is $0 \pmod 4$ and $1 \pmod 3$ is 4, so

$$2^{50^{50}} \equiv 2^4 \equiv \boxed{3} \pmod{13}.$$

Solution 3

To do this, we must compute the expression modulo 100; to do this, we compute it modulo 4 and 25. Clearly, $12^{11^{10}} \equiv 0 \pmod 4$; furthermore, by Euler's Theorem and Problem 5, we have that $12^{20} \equiv 1 \pmod{25}$; furthermore, clearly $11^{10} \equiv (121)^5 \equiv 1 \pmod{20}$, so

$$12^{11^{10}} \equiv 12^1 \pmod{25}.$$

So since the number is $0 \pmod 4$ and $12 \pmod{25}$, we can conclude that it must be $\boxed{12} \pmod{100}$. $\square$

Solution 5

It suffices to instead count the number of n which are not relatively prime to p^α. If a number $m \in \{1, \dots, n\}$ shares a prime factor with p^α, then this shared prime factor must clearly be p. That is, $p \mid m$. So, the set of such m is simply the multiples of p. Clearly, there are $n/p = p^{\alpha-1}$ multiples of p out of the integers from 1 to n; thus, the number of positive integers from 1 to n which are relatively prime to n is

$$n - p^{\alpha-1} = (p-1)p^\alpha,$$

as desired. $\square$

Solution 6,7,8

Note that problems 6,7 are just certain cases of problem 8, so it suffices to just prove the statement of problem 8.

First, we will count the number of positive integers less than or equal to n which are not relatively prime with n. Any such number must clearly be either a multiple of p or q, or of both; so, to count the number of these, we add the number of multiples of p and the number of multiples of q. However, this double counts any number which is a multiple of both p and q, so we subtract the number of multiples of pq.

There are clearly $n/p = p^{\alpha-1}q^{\beta}$ multiples of p in $\{1, \ldots, n\}$; similarly, there are $n/q = p^{\alpha}q^{\beta-1}$ multiples of q, and $n/pq = p^{\alpha-1}q^{\beta-1}$. Hence the number of numbers less than or equal to n not relatively prime to n is $p^{\alpha-1}q^{\beta} + p^{\alpha}q^{\beta-1} - p^{\alpha-1}q^{\beta-1}$.

To get the answer, we subtract this from $n = p^{\alpha}q^{\beta}$, which gives

$$p^{\alpha}q^{\beta} - (p^{\alpha-1}q^{\beta} + p^{\alpha}q^{\beta-1} - p^{\alpha-1}q^{\beta-1}) = (p^{\alpha} - p^{\alpha-1})(q^{\beta} - q^{\beta-1}),$$

upon factoring the expression. □

Solution 9

Consider the equation modulo 81. For $n \geq 9$, we have that $81 \mid n!$. Also, note that by computation,

$$\sum_{i=1}^{8} i! \equiv 63 \pmod{81}.$$

In particular, this means that for $n = 8$, the left side of the equation is $81j + 63$, and as noted above, this will in fact still be some $81j + 63$ for any $n \geq 8$ because we will only add multiples of 81.

But this means that for $n \geq 8$, the equation is $81j + 63 = m^k \implies 9(9j + 7) = m^k$. Note that the highest power of 3 dividing the left side is 3^2, which implies $k \leq 2$ (since $3 \mid m^k \implies 3 \mid m$, so then $k \leq 2$). It is given that $k \geq 2$, so $k = 2$.

Then, the solutions (see problem 5 from week 2) are $(n, m) = (3, 3), (1, 1)$, so the answer is $\boxed{(3, 3, 2), (1, 1, k), k \geq 2}$ (we can easily verify that there exist no solutions with other k for $n \leq 7$). □

Solution 10

First we consider the equation modulo p. By Fermat's little theorem, $3^p \equiv 3 \pmod{3}$, so the equation becomes

$$3 - 4 \equiv m^2 \pmod{p} \implies -1 \equiv m^2 \pmod{p}.$$

Then, note that $(-1)^{\frac{p-1}{2}} \equiv (m^2)^{\frac{p-1}{2}} \pmod{p}$. But the right side simplifies to $m^{p-1} \equiv 1 \pmod{p}$. Hence we have $(-1)^{\frac{p-1}{2}} \equiv 1 \pmod{p}$. This means $\frac{p-1}{2}$ must be even, i.e. $p \equiv 1 \pmod{4}$.

Now, considering the equation modulo 4 gives

$$7(1) + (-1)^p - 4 \equiv m^2 \pmod{p} \implies 7 - 1 - 4 \equiv 2 \equiv m^2 \pmod{4}.$$

Note that $(-1)^p \equiv -1 \pmod{4}$ because p is odd. So $m^2 \equiv 2 \pmod{4}$. But this does not hold for any integer m, so there are in fact no solutions to the equation. □

Solution 11

First note that if $5 \mid x$, then $x^2 \equiv 0 \pmod{25}$. Now we consider all other values of x modulo 25. To count the number of distinct values of x^2, we will see in what cases $x^2 \equiv y^2 \pmod{25}$ holds when $x \neq y$. From factoring this with difference of squares, this implies $25 \mid (x - y)(x + y)$. Note that since $5 \nmid y$, and since the two terms $x - y$ and $x + y$ differ by $2y$, they cannot both be multiples of 5. Hence one of them must be divisible by 25. But $x \neq y$, so we must have $25 \mid x + y$. That is, $x + y = 25$, as we are picking x, y from $\{1, \ldots, 24\}$.

The number of $x \in \{0, \ldots, 24\}$ which are not multiples of 5 is 20, so this contributes 10 distinct perfect squares mod 25. Then, 0 is also a perfect square as noted in the beginning, so there are $\boxed{11}$ perfect squares mod 25 in total. □

Solution 12

The answer is odd n. To show this, first consider any odd n; we must construct a complete residue class satisfying the desired condition. Let the set of a_i with $a_i = i$ for $0 \leq i \leq n-1$ be the complete residue class. Then, the set of $a_i + i$ contains all distinct numbers: if $a_i + i \equiv a_j + j \pmod{n}$, then since each $a_i = i$, this means $2i \equiv 2j \pmod{n}$. But then since n is odd, this simplifies to $i \equiv j \pmod{n}$ (note that this step does not hold when n is even). Since $i, j \in \{0, \ldots, n-1\}$ this implies $i = j$, so we see that the set $\{a_0, a_1 + 1, \ldots, a_{n-1} + n - 1\}$ in fact consists of n distinct numbers, implying that it is a complete residue class.

Also, we must show that even n do not satisfy the condition. Assume for the sake of contradiction that there does some exist some $\{a_i\}$ such that $\{a_i + i\}$ is also a complete residue classes. Then they contain the same elements in some (possibly different) permutation; in particular, this means that the sums of the elements for each set must be equal. That is,

$$\begin{aligned}\sum_{i=0}^{n-1} a_i &\equiv \sum_{i=0}^{n-1}(a_i + i) \pmod{n} \\ \implies 0 &\equiv \sum_{i=0}^{n-1} i \pmod{n} \\ \implies 0 &\equiv \frac{(n-1)n}{2} \pmod{n}.\end{aligned}$$

But note that for even n, $n-1$ is odd so then it is clear that $n \nmid (n-1)\frac{n}{2}$ by considering the power of 2 in each expression. Hence even n cannot satisfy the desired condition. □

Solution 13

The answer is odd n. The solution is analogous to the above. For any odd n, to construct a residue class satisfying the condition, simply take the set of a_i with $a_i = i$ for each $0 \leq i \leq n-1$. Then, the set of a_i contains all distinct numbers due to the following: the i-th number in this set is $i + 3i = 4i$, so if $4i \equiv 4j \pmod{n}$, then since n is odd we can cancel the factor of 4 to obtain $i \equiv j \pmod{n}$. This cannot happen for distinct indices $0 \leq i, j \leq n-1$. So this implies all of the $a_i + i$ are distinct, meaning that $a_0, a_1 + 3, \ldots, a_{n-1} + 3(n-1)$ is a complete residue class.

So, all odd n satisfy the condition. Now we show that even n do not work. Suppose that n satisfies the given condition. Then, there exists a complete residue class $\{a_i : 0 \leq i \leq n-1\}$ such that $\{a_i + 3i : 0 \leq i \leq n-1\}$ is also a complete residue class, so these two sets contain the same elements, in some (possibly different) orders. So, they have the same sum:

$$\begin{aligned}\sum_{i=0}^{n-1} a_i &\equiv \sum_{i=0}^{n-1}(a_i+3i) \pmod n\\ \implies 0 &\equiv \sum_{i=0}^{n-1} 3i \pmod n\\ \implies 0 &\equiv \frac{3(n-1)n}{2} \pmod n.\end{aligned}$$

But for even n, $n-1$ is odd, so it is clear that $n \nmid 3(n-1)\frac{n}{2}$ by considering the power of 2 in each expression. Hence even n cannot satisfy the desired condition. □

Solution 14

No. Suppose $\{a_1, 2a_2, \ldots, (p-1)a_{p-1}\}$ is a complete residue class where $\{a_1, \ldots, a_{p-1}\}$ also is one. Then, since each of these sets are just some permutation of $\{1, \ldots, p-1\}$, we have that

$$\prod_{i=1}^{p-1} ia_i \equiv (p-1)! \equiv -1 \pmod p,$$

where the last step is by Wilson's theorem; but also, note that

$$\prod_{i=1}^{p-1} ia_i \equiv \left(\prod_{i=1}^{p-1} i\right)\left(\prod_{i=1}^{p-1} a_i\right) \equiv (p-1)!(p-1)! \equiv 1 \pmod p,$$

again by Wilson's theorem. So this implies that $-1 \equiv 1 \pmod p$, but this is not possible since $p > 2$. □

Solution 15

By Fermat's Little Theorem, $2^6 \equiv 1 \pmod 7$. Hence it suffices to compute the exponent modulo 6. Note that $1000^{1000} \equiv 4^{1000} \pmod 6$. But note that any power of 4 is just 4 (mod 6), so $1000^{1000} \equiv 4^{1000} \equiv 4 \pmod 6$. This means that

$$2^{1000^{1000}} \equiv 2^4 \pmod 7,$$

by Fermat's Little Theorem. Then clearly $2^4 \equiv \boxed{2} \pmod 7$. □

Solution 16

We compute it separately modulo 3 and 5. Clearly 1000^{1000} is even, so

$$2^{1000^{1000}} \equiv (-1)^{1000^{1000}} \equiv 1 \pmod 3,$$

and similarly, $n = 500 \cdot 1000^{999}$ is even, so

$$2^{1000^{1000}} \equiv 2^{2\cdot 500\cdot 1000^{999}} \equiv (2^2)^n \equiv -1^n \equiv 1 \pmod 5.$$

Hence we can conclude that $2^{1000^{1000}} \equiv \boxed{1} \pmod{15}$. □

Solution 17

The problem is asking us to compute this number modulo 1000. Since $2013 \equiv 13 \pmod{1000}$, we have $2013^{1000} \equiv 13^{1000} \pmod{1000}$. Now, to compute this value, we consider it modulo 8 and 125 separately (since $1000 = 2^3 \cdot 5^3$).

First, note that $13^{1000} \equiv 5^{1000} \equiv 1 \pmod 8$, where the last step is due to the fact that $5^2 \equiv 1 \pmod 8$.

Now, note that by Euler's Theorem and the result of Problem 5 above, $13^{100} \equiv 1 \pmod{125}$. Thus $13^{1000} \equiv 1^{10} \equiv 1 \pmod{125}$. Combining this with the above gives that the number is in fact $\boxed{1} \pmod{1000}$. $\square$

Solution 18

In order to compute this modulo 1000, we compute it modulo 8 and 125. Also, clearly it is equivalent to compute the last three digits of 17^{2017}. First, note that

$$17^{2017} \equiv 1^{2017} \equiv 1 \pmod 8.$$

Also, by Euler's Theorem and the result of Problem 5, we have $17^{100} \equiv 1 \pmod{125}$. Hence $17^{2017} \equiv 17^{17} \pmod{125}$. Now, it is possible to just compute this as

$$17^{17} \equiv 17 \cdot (39)^8 \equiv 17 \cdot (21)^4 \equiv 17(66)^2 \equiv 52 \pmod{125}.$$

Then, combining this with the fact that it is $1 \pmod 8$ gives that the number is $\boxed{177} \pmod{1000}$. $\square$

Solution 19

We must calculate the expression modulo 1000. Clearly, it is equivalent to $3^{2002^{2001}} \pmod{1000}$. Now, note that $2002^{2001} = 2 \cdot (1001 \cdot 2002^{2000})$; for simplicity, let $n = 1001 \cdot 2002^{2000}$; then, we have

$$3^{2002^{2001}} \equiv 3^{2 \cdot n} \equiv (3^2)^n \equiv 9^n \pmod{1000}.$$

Now, by the binomial theorem,

$$9^n = (10-1)^n = \sum_{i=0}^{n} 10^i (-1)^{n-i} \binom{n}{i}.$$

Clearly any term corresponding to $i \geq 3$ will simply be 0 modulo 1000. Hence it suffices to sum the first three terms:

$$9^n \equiv (-1)^n \binom{n}{0} + 10(-1)^{n-1}\binom{n}{1} + 100(-1)^{n-2}\binom{n}{2} \pmod{1000}.$$

Furthermore, note that n is even, so $(-1)^n = 1$; also, we will compute n modulo 1000: note that $n \equiv 1 \cdot 2^{2000} \pmod{1000}$, and to compute this, we separately compute it modulo 8 and 125 (since $1000 = 2^3 \cdot 5^3$).

Clearly, $n \equiv 2^{2000} \equiv 0 \pmod 8$. Furthermore, by Euler's Theorem and the property shown in problem 5, $2^{100} \equiv 1 \pmod{125}$, implying that $2^{2000} \equiv 1 \pmod{125}$. Combining these gives that $n \equiv 376 \pmod{1000}$.

We must compute

$$9^n \equiv (-1)^n \binom{n}{0} + 10(-1)^{n-1}\binom{n}{1} + 100(-1)^{n-2}\binom{n}{2} \equiv 1 - 10n + 100n(n-1)/2 \pmod{1000},$$

and we know $n \equiv 376 \pmod{1000}$, so now computation gives $9^n \equiv \boxed{241} \pmod{1000}$. $\square$

Solution 20

The answer is n. A short proof is the following: consider the fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ when they are simplified. Clearly, there are n fractions in total. But also, note that for each d dividing n, $\phi(d)$ of those fractions have a denominator of d when written in simplified form. Furthermore, it is clear that each of the fractions can be expressed as a simplified fraction $\frac{k}{d}$ for some k, d with $d \mid n$. Thus, the number of fractions $\frac{1}{n}, \ldots, \frac{n}{n}$ is also equal to $\sum_{d|n} \phi(d)$, so $\sum_{d|n} \phi(d) = n$. $\square$