Week 1: Information Theory

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Problem 1.
(a) A magician guessed an integer number between 1 and 100. You can ask him any Yes/No questions about the number. Can you guess the number with full certainty in 7 questions?

(b) Can you do this in 6?

Watch this video by 3Blue1Brown on solving Wordle using information theory.

Problem 2.
(a) You are given a set of scales and 3 coins. The scales are of the old balance variety. That is, a small dish hangs from each end of a rod that is balanced in the middle. The device enables you to conclude either that the contents of the dishes weigh the same or that the dish that falls lower has heavier contents than the other. The 3 coins appear to be identical. In fact, 2 of them are identical, and one is counterfeit and is lighter than the other 2. How do you determine which coin is counterfeit in 1 weighing?

(b) Same problem with 27 coins and 3 weighings.

(c) Can you determine a counterfeit coin among 28 coins in 3 weighings?

Problem 3.
Suppose you got the following first guess in Wordle: ("f" is grey and "ight" are green).

(a) List all the words matching this pattern (you should get eight words, check with the instructor).

(b) Can you guess the word in 5 remaining attempts?

(c) [*] How many extra attempts do you need to guess it for sure?

Problem 4.
You are given 9 labeled marbles, of which 2 are radioactive. For any set of marbles, in one check you can find out if it contains at least one radioactive marble (but you cannot find out how many there are).
(a) Count the number of possibilities for the identity of the two radioactive marbles. (e.g. radioactive marbles are marble 1 and marble 5, or marble 2 and 7, etc.)

(b) How many checks do you need to do to find the two radioactive marbles?

(c) [*] Find an algorithm guaranteeing this number of checks.

Problem 5.

You are given a set of scales and 12 marbles.

The 12 marbles appear to be identical. In fact, 11 of them are identical, and one is of a different weight. Your task is to identify the unusual marble and discard it. You are allowed to use the scales three times if you wish, but not more.

Note that the unusual marble may be heavier or lighter than the others. You are asked to both identify it and determine whether it is heavier or lighter.

Hint: You should distinguish between 24 possibilities. Compose your first weighing so that each of the three outcomes is equally likely.

Problem 6 (*)**.**

There are 5 weights of different mass. In one operation, one can choose an ordered triple of weights (A, B, C) and find out if the statement "$m(A) < m(B) < m(C)$" is true (where $m(X)$ denotes the mass of the weight X). Is it possible to use 9 such operations to find out the ordering of the weights?