MATH CIRCLE WINTER 2017.
SURFACE DECOMPOSITIONS

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Surfaces A *surface* is an object that, near any given point, looks like the plane - examples include a disc, a sphere, and the surface of a donut. For today's purposes, all of our surfaces will be ones that can sit nicely in 3d space (no Klein bottles), and have a defined inside and outside (no Mobius bands) - but we will allow boundaries, as long as the boundary is a finite set of circles (so we're allowing discs and annuli and so on.)

Definition (Genus)**.** *The* genus *of a surface is the maximum number of non-intersecting circles on its surface that we can cut along while the surface remains connected.*

Problem 1. *Give examples of surfaces of genus 1, 2, and 4.*

Problem 2. *Give examples of genus 0 surfaces with 0, 1, 2, and 3 boundary circles.*

1. Decomposing

Definition.
- *A* cut *of a surface is precisely that - cutting the surface along a closed loop.*
- *An* n-decomposition *of a surface is a collection of disjoint cuts such that, if you make all of the cuts, the surfaces you are left with are all genus 0 with n boundary circles.*

Problem 3.
(1) *Which surfaces have a 0-decomposition? How many different 0-decompositions do they have?*

(2) *Which surfaces have a 1-decomposition? How many different 1-decompositions do they have?*

(3) *Which surfaces have a 2-decomposition? How many different 2-decompositions do they have?*

2. 3-Decompositions

For the rest of this handout, we'll be considering the problem of 3-decompositions.

Problem 4. *Give an example of a surface with no boundary circles that has a 3-decomposition.*

Problem 5. *For which values of k can you find surfaces with no boundary circles that have a 3-decomposition into k pieces? What is the relationship between k and the genus of the resulting surface? (Hint: think about going in reverse - starting with k surfaces of genus 0 with 3 boundary circles, and attaching those boundaries together until none are left.)*

Definition. *A* tiling *of a surface is a collection of edges and vertices on it and its boundary, so that if you cut along them, you get a collection of (potentially curved) polygonal pieces.*

Problem 6. *Give an example of tilings for several surfaces. Keep in mind that all segments have to start and end at vertices, and the entire boundary (if it exists) has to be covered by the segments.*

Definition. *If we have a tiling of a surface, the* Euler characteristic *of that tiling is defined as* $V - E + F$*, where* V *is the number of vertices in the tiling,* E *is the number of edges, and* F *is the number of faces.*

Problem 7. *Show that the following changes to a tiling don't change its Euler characteristic:*

- *Splitting an edge into two with a new vertex in the middle*
- *Splitting a face into two with a new edge in the middle*
- *Joining two edges or faces (i.e., reversing the above processes)*

Problem 8. *Challenge Show that any two tilings of the same surface have the same Euler characteristic (hint: try overlaying them.) Conclude that the Euler characteristic is actually a property of surfaces, rather than tilings.*

Problem 9. *Find the Euler characteristic of a genus 0 surface with 3 boundary circles.*

Problem 10. *Prove that if a cut breaks a surface into two pieces, the Euler characteristic of the original is the sum of the Euler characteristics of the pieces.*

Problem 11. *What is the Euler characteristic of a genus g surface? Use this to say how many pieces there could be in a 3-decomposition of a genus g surface.*

Problem 12. *Classify $n-$decompositions of surfaces without boundaries, for $n \geq 4$.*