# ORIGAMI CONSTRUCTIONS II

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In this problem set we will continue to explore Origami constructions. We will finish the comparison between classical and Origami constructions and we will see that cube roots and angle-trisectors can be constructed by folding. Recall that the allowed steps for Origami constructions were:

1. Intersect two lines.
2. Given two intersecting lines, bisect their angle.
3. Given two points $P$ and $Q$, find the perpendicular intersector of $PQ$.
4. Given a point $P$ and a line $l$, find the line perpendicular to $l$ going through $P$.
5. Given two points $P$ and $Q$ and a line $l$, find the line through $Q$ that reflects $P$ onto $l$.
6. Given two points $P$ and $Q$ and two lines $l$ and $l'$, find a line that reflects $A$ onto $l$ and $B$ onto $l'$.

Furthermore we will need the following definition:

**Definition 1.** Given a line $l$ and a point $F$, the parabola $p$ with focus $F$ and directrix $l$ is the set of points $P$ that is equidistant to $l$ and $F$. (Recall that the distance from $P$ to $l$ is the distance of $P$ to $P'$ where $P'$ is the foot of the perpendicular line through $P$ to $l$.

**Problem 1.** Draw a sketch of a parabola $p$, its focus $F$, directrix $l$, a point $P$ on $p$, and the point $P'$ described above.

**Problem 2.** Calculate the focus and directrix of the parabolas $y = ax^2$ and $x = by^2$. Draw a sketch!

**Definition 2.** Let $P$ be a point on the parabola $p$. A line $l'$ is a tangent to $p$ at $P$ if it only shares one point with $p$ and does not cut $p$.

**Problem 3.** Let $P$ be a point on a parabola with focus $F$ and directrix $l$. Let $P'$ be the foot of the perpendicular to $l$ through $P$. Let $l'$ be the perpendicular bisector of $P'F$.

1. Draw a sketch.
2. Argue that $P$ must be on $l'$.
3. Let $Q$ be a point on the opposite side of $l'$ than $F$. Show that $Q$ cannot be on $p$.
4. Argue that $l'$ must be the tangent to $p$ at $P$.

**Problem 4.** Show that step 5 of the basic folding constructions amounts to finding a tangent to a parabola through a given point. Show that step 6 amounts to finding a common tangent to two different parabolas.

**Problem 5.** Do step 5 of the folding construction using classical methods. *Hint:* If $Q$ is on the tangent to a parabola $p$ at $P$, then we have $QP' = QF$, where $F$ is the focus and $P'$ is the foot of the perpendicular to the directrix $l$ through $P$.

**Problem 6.** So far when folding we never were allowed to use the edges of the paper as marks. Argue that with what you learned last week, it is legitamite to use the edges and corners of a rectangular or square piece of paper as reference point when folding.

**Problem 7.** Start with a square with corners (counterclockwise) $A, B, C, D$. Let $P, R$ and $Q, S$ divide the sides $AB$ and $DC$ into thirds. Fold such that $C$ falls onto $AB$ (call this point $C'$) and $S$ falls onto $PQ$ (call this point $S'$). Let $T$ be the intersection of the crease with $BC$. Assume $C'B = 1$ and call $AC' = x$ and $BT = y$.

1. Sketch and execute the above construction!
2. Express $C'T$ in terms of $x$ and $y$. Remember that $ABCD$ is a square.
3. Verify that
$$y = \frac{x^2 + 2x}{2x + 2}.$$
4. Argue that the triangles $PCS$ and $BTC$ are similar and use this to get another expression for $y$ in terms of $x$. *Solution:*
$$y = \frac{2x^2 + x - 1}{3x}.$$
5. Calculate $x$.
6. What does this enable you to construct by folding. How?

**Problem 8.** In the below sketch the lines $AB$ and $CA$ are orthogonal. The line $k$ is constructed such that it reflects $C$ to $C'$ on the ray $AD$ and $A$ to $A'$ on the perpendicular bisector of $A'$ (this defines $E$ as well). Furthermore $E'$ is the reflection of $E$ on $k$.

1. Show that the triangles $AA'C$ and $A'AC'$ are isoceles.
2. Show that the angles $C'AE'$, $E'AA'$, and $A'AB$ agree.
3. Use this to trisect the angle $DAB$ by folding.