# Wallpaper and Symmetries
Los Angeles Math Circle

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## Rigid Transformations of the Plane

Points $A, B, C$ are called *collinear* if they all lie on a single line, and *noncollinear if they don't*. A *mapping* of the plane is a function which sends points in the plane to points in the plane.

A *rigid transformation* or *motion* of the plane is a mapping which preserves the distances between points.

The following is a key fact about rigid transformations:

> A rigid transformation of the plane is determined by its action on any triangle (i.e. any three noncollinear points).

What examples can you think of? Can you enumerate all possibilities?

## Symmetries

What do we mean when we say that an object is symmetric? The most familiar kinds of symmetry are *reflection symmetry*, and *rotational symmetry*.

In mathematics we commonly expand the meaning of the word to the following: A *symmetry* of an object is any rigid transformation that maps the object perfectly onto itself.

## Problems

1. (a) Name your two favorite rigid transformations.

   (b) Show the effect of each of your favorite transformations on the figure below:

2. Which capital letters of the English language have ... axes of symmetry?
   ... rotational symmetry? (Assume that features like serifs do not matter.)

3. Prove that if a triangle has two axes of symmetry, then it has at least three axes of symmetry.

4. Can a pentagon have exactly two axes of symmetry?

5. It is known that some figure in the plane coincides with itself upon rotation by 48 degrees about a certain point $P$. Is it necessarily the case that this figure coincides with itself after a rotatioin of 72 degrees around $P$?

6. If two triangles have the same side lengths $a, b, c$, prove that one triangle can be moved so that it coincides with the other.

7. Given two circles with equal radius, prove that one can be moved onto the other by means of a rotation. (What should be true of the point should you rotate about?)

8. (a) Prove that a translation can be represented as the composition of two reflections in a point.

   (b) What is the composition of two translations? of two rotations? a translation and a rotation?

   (c) Consider the motion which is a reflection in a line $\ell$, composed with a translation in a direction parallel to $\ell$. Prove that this is not a translation, rotation, or line reflection.

# Motions of the Plane

We have discussed three types of motions of the plane – translations, rotations, and reflections.

To these we must add a fourth kind of motion – glide reflection:

(Exercise 8 in the last section asked you to prove that a glide reflection is not the same as a translation, rotation, or reflection.)

# Classifying Motions of the Plane

**Theorem.** *Every motion of the plane is a translation, rotation, reflection, or glide reflection.*

We will not write out a complete proof here, but the needed ingredients are the following facts:

- Any movement of a triangle to a congruent triangle is achievable by composing the above types of motions.
- A motion is completely determined by its action on any triangle (three noncollinear points).
- Any composition of the above types of transformations is again one of the above types.

These three statements together imply that every motion of the plane is one of the four given types as follows:

- Given any motion of the plane, consider its action on a triangle (any triangle you want).
- Achieve this motion by composing motions of the four kinds above, which is possible by statement 1.
- The composition is the same as the given motion because they do the same thing to the triangle by statement 2.
- The composition is equal to one of the four given types by statement 3.

Another theorem of interest is the following:

**Theorem.** *Every rigid transformation of the plane is the composition of at most three reflections.*

# Problem Solving with Motions of the Plane

Motions of the plane can be helpful in solving the following problems:

1. Given a line $\ell$ and points $A, B$ not on the line, with line $AB$ not parallel to $\ell$, ...
   (a) Find the point $M$ on $\ell$ such that the sum of the distances to $A, B$ is minimal.
   (b) The difference between the distances to $A, B$ is maximal.
2. Given a point $O$ contained in the interior of triangle $ABC$, construct a line segments with endpoints on the perimeter such that $O$ is the midpoint of the line segment.
3. Given a point in the plane and two parallel lines, none of which intersect, find an equaliteral triangle with one vertex at the point, and one vertex on each of the two lines.
4. In trapezoid $ABCD$, where $AD \parallel BC$, points $M, N$ which are the midpoints of $AB$ and $CD$. If the line $MN$ intersects the lines $AB$ and $CD$ in equal angles, prove that the trapezoid is isosceles (i.e. the non-parallel sides have the same length).
5. Given five straight lines, show how to inscribe a pentagon in a circle such that the sides of the pentagon are perpendicular to the given lines.

# Wallpaper symmetry patterns

**Exercise:** Examine the various wallpaper patterns that have been given to you. Try to identify which of the following are present.

- Translation symmetries
- Lines of reflection symmetry
- Points of rotational symmetry
- Unaccounted-for glide reflection symmetry

**Equivalent points:** When a pattern has symmetries, we often "identify" each point with all points related to that point by a symmetry. Bear this in mind when we start to count up features of a symmetry pattern.

The set of all points equivalent to a given point is called that point's *orbit*.

**Features:** We give names to several types of feature:

- Kaleidoscope ($*$): A meeting of lines of symmetry
- Gyration ($\backsim$): A (non-kaleidoscope) point of rotational symmetry
- Miracle ($\times$): Equivalent points joined by a path not crossing a line of symmetry ("mirrorless crossing")
- Wonder ($\circ$): The absence of any of the above

The *signature* of a symmetry pattern is determined as followed:

- For each $n$-fold gyration, write an $n$.
- If there are any kaleidoscopes, write a $*$.
- For each $n$-fold kaleidoscope, write an $n$.
- For each miracle, put a $\times$.
- If none of the above happens, put a $\circ$.

**Exercise:** Try to identify all the above "features" and write the signature of various wallpaper patterns.

# Why 17 wallpaper patterns?

**Theorem.** *The signature of each symmetry pattern has characteristic value 2.*

From here it is a short matter to enumerate all the possible signatures by trial and error:

$$632, \quad *632, \quad 442, \quad 442, \quad *333, \quad 333, \quad *2222, \quad 2222$$
$$4*2, \quad 3*3, \quad 2*22, \quad 22*, \quad 22\times, \quad **, \quad *\times, \quad \times\times, \quad \circ$$

## References


J. Conway, H. Burgiel, C. Goodman-Strauss, *The Symmetries of Things.* CRC Press, 2008.

M. Senechal, *Crystalline Symmetries: An informal mathematical introduction.* Taylor & Francis, 1990.

Inkscape: `https://inkscape.org/`

Kali: `http://www.geometrygames.org/kali`