TOPOLOGY AND SURFACES

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1. Concepts of Topology

Topology, sometimes called “rubber-sheet geometry,” is the part of math where we study geometric properties that hold true even when space is stretched, squished, or bent. As long as you don’t tear or glue a space, then topologically speaking, it’s the same as before.

In particular, today we’ll be looking at spaces that, if you zoom in very close, look the same as the plane, because 2 dimensions is the lowest-dimensional case where interesting things happen (for 1 dimension, every space that looks like the line is either a line or a circle).

Problem 1 What’s an example of a topological property? What’s a property that isn’t topological?

Problem 2 Suppose you have houses A, B, and C, and utilities X, Y, and Z. Each house needs to be connected to every utility, and the connecting lines cannot cross.

- Can you do this on a flat plane?

- Can you do this on a torus (the surface of a donut)?

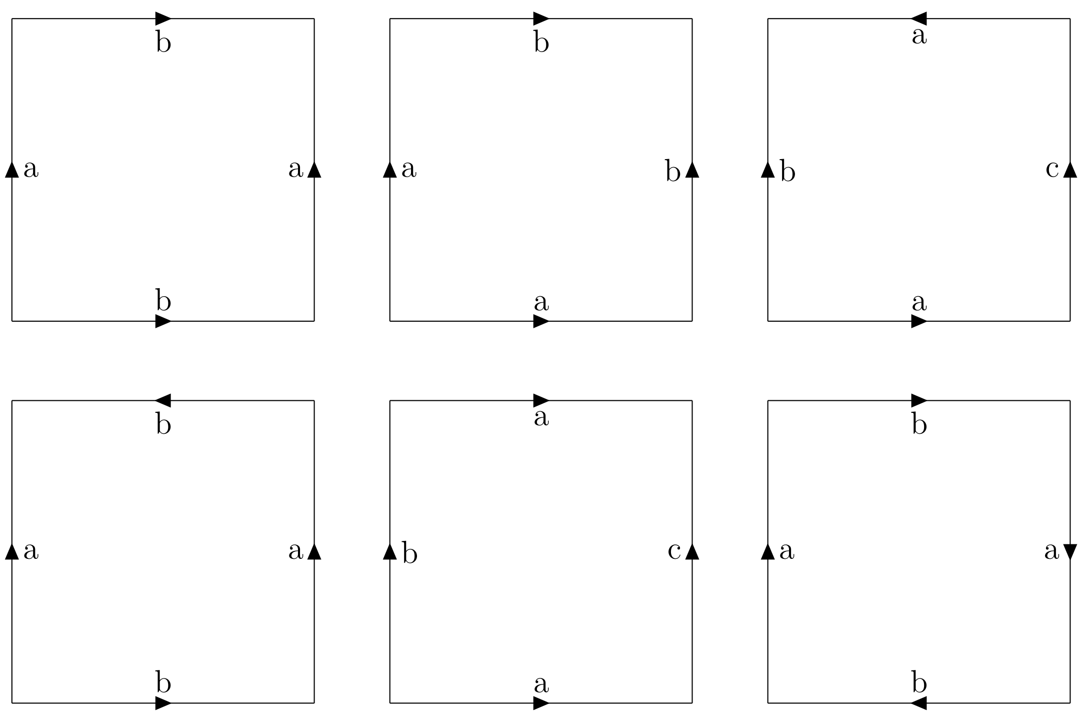
2. Surfaces and Polygonal Presentations

We're going to be studying surfaces, but there's an issue: this handout exists in only 2 dimensions. Most surfaces aren't subsets of the plane. In fact, we'll even see some surfaces that aren't subsets of 3 dimensional space. So, we need a way to represent them. The tool we'll use for that is polygonal presentations.

Definition A **polygonal presentation** is a polygon with labeled, directed sides, corresponding to the surface you get by attaching the sides with the same label, with the arrows matching up when they are attached.

Let's look at some examples.

Problem 3 Here are six polygonal presentations; one of them is a sphere, one is a torus, one is a circular band, one is a klein bottle, one is a mobius band, and one is projective space (the space you get by identifying opposite points on a sphere). Figure out which is which!

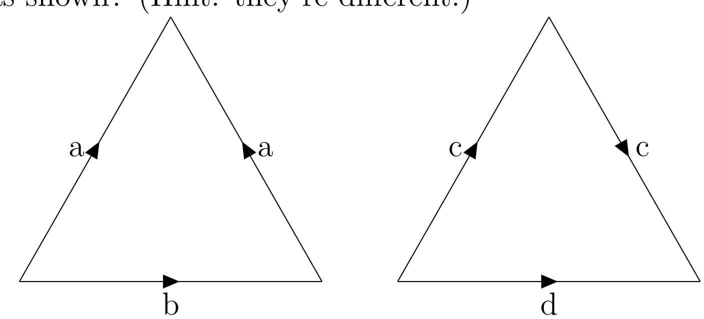


Use the polygonal presentations to answer the next questions.

Problem 4 What do you get if you cut a mobius band down the middle circle?

Problem 5 What do you get if you cut a klein bottle in half? Does it matter which direction you cut?

Problem 6 What two surfaces do you get by gluing the edges of each of these triangles as shown? (Hint: they're different.)



3. Tic-Tac-Toe

Problem 7 Play some tic-tac-toe on a torus (use the square polygonal presentation, and keep in mind that now there are 6 diagonals, not just 2). Is a cat's game possible?

Problem 8 How many essentially different first moves are there for tic-tac-toe on a torus? How many essentially different second moves are there?

Problem 9 Is there a winning strategy for tic-tac-toe on the torus? Who wins? Does your answer change if the first player has to go on an edge space?

Problem 10 Play some tic-tac-toe on a Klein bottle. Can you get a cat's game? Does the first player have a winning strategy?

4. Connected Sums

Definition The **connected sum** of two surfaces is obtained by cutting out a disk from each, and attaching them along the boundary circles.

Problem 11 What do you get when you take the connected sum of two projective planes? (Hint: first figure out what you get when you remove a disk from a projective plane. Then figure out what you get from putting two of those together.)

Problem 12 Find a polygonal presentation for a two-holed torus, the surface you get by taking the connected sum of two tori.

Problem 13 Show that the connected sum of a torus and projective plane is the same as the connected sum of a Klein bottle and a projective plane.