Games with Payoffs

1. [Rock-Paper-Scissors variant] Consider modifying a game of rock-paper-scissors so that player 1 cannot use scissors and player 2 cannot use rock.

(a) Try 10 games with a neighbor. Write down your observations. Do you think one player has an advantage here? Which one? Come up with a strategy as each player and test them out.

(b) Say a win gives 1 point, a draw gives 0 points, and a loss gives -1 points. Write down a payoff matrix for this game.

$$\begin{pmatrix} (\quad , \quad) & (\quad , \quad) \\ (\quad , \quad) & (\quad , \quad) \end{pmatrix}$$

(c) If both players play completely randomly, what is the expected value of player 1's score after 1 round of the game? player 2's score? What about after 10 rounds?

(d) Suppose player 1 chooses rock with probability p (and thus paper with probability $1 - p$) and player 2 chooses paper with probability q (and thus scissors with probability $1 - q$). What is the expected value of player 1's score after 1 round? Make sure your answer is consistent with part (c).

(e) Factor your answer to part (d) in the form $c(p - a)(q - b) + d$ for some constants a, b, c, d. Based on this, what is the largest expected value player 1 can guarantee by choosing p without knowing how player 2 will choose q?

(f) By similar reasoning, find an optimal value of q for player 2. Compare the expected value of each player's score with these strategies to part (c). Does one player have more of an advantage with this careful analysis?

2. [Rendezvous Search] Two friends travelling independently to Westwood want to meet for lunch when they arrive. One friend remembers a Subway there, and they decide to meet there. When they arrive, they realize there are actually $n > 1$ Subways in Westwood and record the locations. Then both friends' phones run out of power, so they cannot communicate. They initially head to two different Subways and realize the problem. Assuming they still want to meet as quickly as possible at a Subway, what should they do?

(a) Consider the case $n = 2$. Assume the players must use the same strategies and make a travel decision every twenty minutes.

Here's a possible strategy: Stay at your Subway with probability $1/3$, and head to the other Subway with probability $2/3$. Note if both decide to head to the other Subway, we assume they miss each other on the way to keep the problem interesting. What's the expected probability they meet up after the first decision and (possible) travel takes place? After 3 rounds of this?

(b) What's the probability of eventually meeting (over infinitely many decision and travel turns) if both players use this strategy? You will have to compute an infinite series (remember $a+ar+ar^2+\cdots = a/(1-r)$ for $r < 1$).

(c) What is the expected number of turns before they meet with this strategy?

(d) Can you come up with a better strategy for the $n = 2$ case? What is the expected number of turns for your strategy? (Hint: To compute the infinite series S, consider computing $S - rS$ first.)

[Challenge] Devise a couple strategies for $n = 3$. Which is best?

3. Suppose two people are talking on the phone and want to flip a coin to decide something, but can't trust each other to tell the truth about the flip? Assume they can't access a third person or other source of arbitration to flip the coin and communicate it to them. How can they approximately flip a coin and trust its result?

4. Suppose there are five people, each of which has a secret number between 0 and 100, and we want to find the average of all these numbers without revealing any of their values. Again, some outside party is not allowed to record values.

Here's one potential scheme: Some first person takes a number $M > 1000$ (which only they know), then adds their secret number to it, and passes it around, and everyone adds their number to it. When it comes back around, the first person subtracts M, and divides by 5 to find the average.

(a) Do you see any problems with this scheme? Can anyone find anyone else's secret number on their own? What if players three and five work together? What can they discover?

(b) [Challenge] See if you can use random numbers in the set $\{1, 2, \ldots, M\}$ somehow to fix this problem. There is a (tricky) way to make it so the average is computed but no group of three of fewer people can figure out someone's secret number.