RAMSEY THEORY

LAMC HIGH SCHOOL 1

1. Ramsey Numbers

This week, we'll be looking at *Ramsey numbers*, which are defined via a series of increasing generalizations.

Definitions

- The *edge Ramsey numbers*: $R(m, n)$ is defined to be the minimum integer such that, if $N \geq R(m, n)$, and each edge of K_N is colored red or blue, then there must either be a red monochromatic K_m or a blue monochromatic K_n subgraph. This is also written $R_2(m, n)$, with the "2" referring to the fact that we're coloring edges, but if there's no subscript, we assume that it is 2 (yesterday, we showed that $R(3, 3) = 6$).
- The *Multi-Ramsey numbers*: Now, call a subset a $u-$subset if it contains u elements. Then, define $R_u(m, n)$ to be the minimum integer such that, if $N \geq R_u(m, n)$, and each $u-$subset of a set with N elements is "colored" red or blue, then there must either exist an $m-$subset, all of whose $u-$subsets are red, or an $n-$subset, all of whose $u-$subsets are blue.
- The *Ramsey numbers*: the last generalization is to allow any number of colors, and define $R_u(n_1, n_2, \dots, n_k)$ as above, where k is the number of distinct colors.

Problem 1 What is $R_1(2, 2, \dots, 2)$, where there are k 2's inside the parentheses?

Problem 2 Suppose there are 17 students at a math circle meeting, and for each pair of students, exactly one of the following is true: "they haven't met", "they are good friends," or "they are related." Prove that there exist a group of 3 students, all of whom are either mutual strangers, mutual good friends, or family. In particular, phrase this proof as a statement in terms of Ramsey numbers.

Problem 3 Ten people are locked in a vault hiding from zombies. Prove that either three know each other, or four are mutual strangers; show that this is not necessarily true if we only have 8 people. Then, phrase these results in terms of Ramsey numbers.

Problem 4 Color every lattice point (point with integer coordinates) either red or blue. Prove that there must be a rectangle, whose sides are parallel to the axes, with all four vertices monochromatic.

Problem 5 Prove that, for all positive integers m, n, $R(m, n) \leq R(m, n-1) + R(m-1, n)$ (hint: induct on $m + n$).

Problem 6 Use the previous problem to prove the first Ramsey's Theorem: $R(m, n)$ is defined and finite for all m, n.

Problem 7 Using these recurrence relations, and constructions as needed, show that

(1) $9 \leq R(3,4) \leq 10$
(2) $14 \leq R(3,5) \leq 15$

Problem 8 Prove that, for all positive integers m, n, u, $R_u(m,n) \leq R_{u-1}(R_u(m,n-1), R_u(m-1,n)) + 1$.

Problem 9 Use the previous problem to prove the second Ramsey's Thorem: $R_u(m,n)$ is defined and finite for all m, n, u.

Problem 10 Prove that, for all $u, n_1, n_2, \ldots, n_k$, $R_u(n_1, n_2, \ldots, n_k) \leq R_u(R_u(n_1, \ldots, n_{k-1}), n_k)$.

Problem 11 Use the previous problem to prove the full version of Ramsey's Theorem: Ramsey numbers always exist, and are finite.

2. Erd?s-Szekeres Theorem

Problem 12 Prove that if you have 5 points in the plane, no three of which are collinear, some 4 of them are the vertices of a convex quadrilateral (Hint: look at the convex hull - the set of points between any pair of your 5 points).

Definition Define the $n-$gon number $N(n)$ to be the least number such that, if $N \geq N(n)$, any set of N points in the plane, with no three collinear, have an $n-$subset which are the vertices of a convex $n-$gon.

Problem 13 Show that the $n-$gon number $N(n) \leq R_4(n, 5)$ as follows: color the 4-sets of points in the plane the color 1 if they make a convex quadrilateral, and color 2 if they don't make a convex quadrilateral. Use the previous problem to rule out a monochromatic color-2 subset of size 5; thus, there must be a monochromatic color-1 subset of size n; then explain why this must be convex.

3. Van Der Waerden's Theorem

One way to further develop Ramsey theory is to put restrictions on the subsets allowed to be considered.

Definition For each $k \geq 1$ and $c \geq 1$, there exists n such that for every c-coloring of $\{1, 2, \ldots, n\}$, there exist $a, d, d \neq 0$ such that $\{a, a+d, \ldots, a+(k-1)d\}$ is monochromatic. We call this number $W(k, c)$.

Problem 14 Prove that $W(3, 2)$ exists, and is finite.