QUATERNIONS & ROTATIONS

LAMC HIGH SCHOOL 1

1. Inverting Quaternions

Last week, we introduced the Quaternion numbers, $\mathbb{H}$, which are a real algebra generated by $1, i, j, k$, with the multiplication relations:

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k, jk = i, ki = j$$

$$ji = -k, kj = -i, ik = -j$$

Definition 1 We can view $\mathbb{H} \cong \mathbb{R} \times \mathbb{R}^3$ where the first term is the real part of quaternions, generated by 1, and the $\mathbb{R}^3$ term is generated by i, j, and k. We can then write a quaternion $q \in \mathbb{H}$ as $q = a + x$, where $a \in \mathbb{R}$ and $x \in \mathbb{R}^3$.

- Let $q = a + x \in \mathbb{H}$. Then $a = \mathcal{R}(q) \in \mathbb{R}$ is the *real part* of q, and $x = \mathcal{I}(q) \in \mathbb{R}^3$ is the *imaginary part* of q.
- The complex conjugate of $q = a + x \in \mathbb{H}$ is defined as $\overline{q} := a - x \in \mathbb{H}$
- The norm of $q \in \mathbb{H}$ is defined as $N(q) := q\overline{q}$

Now let's use these ideas to show that every non-zero quaternion has a multiplicative inverse.

Problem 1: Show that $N(a + x) = |a|^2 + |x|^2$, where $|a|$ is the absolute value, and $|x|$ is the standard (Euclidean) norm.

Problem 2: Show that $\overline{pq} = \overline{q}\,\overline{p}$

Problem 3: Use the previous problems to show that $N(pq) = N(p)N(q)$.

Problem 4: Find the inverse to any non-zero quaternion $q \in \mathbb{H}$ (Hint: how do you do this in $\mathbb{C}$?)

2. Scalar Products

Definition 2 Let $x, y \in \mathbb{R}^3$; we can view them as strictly imaginary quaternions, by viewing $x = (x_1, x_2, x_3)$ as $x_1 i + x_2 j + x_3 k \in \mathbb{H}$ and $y = (y_1, y_2, y_3)$ as $y_1 i + y_2 j + y_3 k \in \mathbb{H}$. Then we define the *scalar product* as $x \cdot y := -\mathcal{R}(xy)$.

Problem 5 Find an explicit formula for the scalar product of two strictly imaginary quaternions.

Problem 6 Convince yourself that x and y are perpendicular in $\mathbb{R}^3$ if and only if $x \cdot y = 0$.

Problem 7 Prove that $x \cdot y = 0$ if and only if $x\overline{y} = -y\overline{x}$.

Problem 8 Show that $xx = -x \cdot x = -|x|^2 = -N(x)$

Problem 9 Let $u, v \in \mathbb{R}^3$ be perpendicular unit vectors, and define $w := uv$.

- Show that w is both strictly imaginary ($\mathcal{R}(w) = 0$) and also a unit quaternion ($N(w) = 1$).

- Verify that $u^2 = v^2 = w^2 = uvw = -1$

- Check that any purely imaginary Quaternion is a sum of real multiples of u, v, and w.

3. Relating Quaternions to Rotation

Problem 10 Let q be a unit quaternion. Show that we can wrute $q = cos(\theta) + sin(\theta)u$ for some angle θ and some unit, strictly imaginary quaternion u.

Let $x \in \mathbb{R}^3$ be any point, and let $q = cos(\theta) + sin(\theta)u$. We want to show that the point $qx\overline{q}$ is the point in $\mathbb{R}^3$ that results from rotating x by an angle of 2θ around the axis through 0 and u.

Problem 11 Check that the above is true if u is i, j, or k.

Problem 12 Explain why it's true for any axis u, by replacing i, j, and k with the axes u, v, and w from before.

Problem 13 Show that the composition of two rotations in $\mathbb{R}^3$ is another rotation.

Problem 14 Let A by the rotation in $\mathbb{R}^3$ around the i-axis by angle θ, and let B be the rotation around the j-axis by angle ϕ. Calculate the axis of the rotation AB (rotating by B, then by A).

Problem 15 Is it possible to define a similar multiplicative structure on $\mathbb{R}^3 \cong \mathbb{R}^1 \times \mathbb{R}^2$.