COMBINATORIAL GAME THEORY
Handout 2 - Nim

High School 1
Los Angeles Math Circle

Rules of nim

- We play with various piles of beans.
- On their turn, a player removes one or more beans from the same pile.
- The first player who cannot move loses.

Exercise 0.1. *Play a few rounds of nim.*

Definition. Given two natural numbers a and b, their *nim-sum*, written $a \oplus b$, is calculated as follows:

- write a and b in binary,
- "add them without carrying", i.e., for each position digit,
  $0 \oplus 0 = 1 \oplus 1 = 0, \qquad 0 \oplus 1 = 1 \oplus 0 = 1.$

Example: $5 \oplus 6 = ?$

$$\begin{array}{rcccc} & & \text{in binary} & & \\ 5 & = & 101 & & \\ 6 & = & 110 & & \\ 5 \oplus 6 & = & 011 & = & 3 \end{array}$$

Exercise 0.2. *Prove the following theorem:*

Theorem. *A position (a, b, c) in nim is a P-position if and only if $a \oplus b \oplus c = 0$.*

Proof. (You will need to fill in the details.) To prove this, we need to check three things:

1. If we start in a position with nim-sum non-zero,
   then we can always move into at least one position with nim-sum zero.
2. If we start in a position with nim-sum zero,
   then we cannot move into another position with nim-sum zero.
3. The final position $(0, 0, 0)$ has nim-sum zero.

□

More Nim Problems

Exercise 0.3. *What would be your move in nim if you start from the following position?*

1. $(60, 70, 81)$
2. $(5, 6, 7, 8, 9)$

Exercise 0.4. *In preparation for next week's class, let's combine subtraction games and nim. We now play with various piles of beans. A legal move is to remove 1, 2, 3, or 4 beans from the same pile. The first player who cannot move loses. What is your strategy?*

Exercise 0.5. *Try to solve the previous problem with* $\{1, 3, 4\}$ *instead of* $\{1, 2, 3, 4\}$.

*[*Note: *This question will be very hard right now, but much easier at the end of this week. If you have no idea how to do it, just wait a couple of days.]*