Math Circle Advanced 1

Winter quarter 2022 Final game

1 Algebra

Problem 1.1.
Let $P(n)$ be a polynomial of degree 3 such that $P(0) = 1$, $P(1) = 4$, $P(2) = 11$ and $P(3) = 28$. What is $P(4)$?

Problem 1.2.
Write 4 algebraic numbers that are not rational numbers, and explain why they are algebraic and irrational.

Problem 1.3.
Find $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}$.

Problem 1.4.
Find all integers n such that $\frac{n^2+3}{n-1}$ is an integer.

Problem 1.5.
Find a polynomial with integer coefficients that has $\sqrt{5} - \sqrt[3]{2}$ as a root.

2 Geometry

Hint: It will be a good idea to draw these problems when solving them.

Problem 2.1.
Let ABC be a triangle such that point B lies inside the circle with diameter AC. What is the smallest possible value for $\angle ABC$?

Problem 2.2.
Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that EF is perpendicular to DB, and A and C lie on DE and DF, respectively. What is the length of EF?

Problem 2.3.
The diagonals of convex quadrilateral $ABCD$ meet at point P. The area of the triangles ABP, BCP, CDP is equal to $2, 3, 4$ respectively. Find the area of triangle ADP.

Problem 2.4.
A point E is taken on side AC of the triangle ABC. Through E pass straight lines DE and EF parallel to sides BC and AB, respectively; where D and E are points on AB and BC, respectively. Suppose that the area of the triangle ADE is 4, and the area of the triangle EFC is 3. Find the area of the quadrilateral $BDEF$.

Problem 2.5.
Let ABC be a triangle, and let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Suppose that we have $3\alpha + 2\beta = 180°$, and that both AC and BC have length 1. Compute the length of AB.

3 Combinatorics

Problem 3.1.
A country has 100 cities and some roads connecting them. Every road connects 2 cities and every city has 5 roads going from it. What is the total number of roads?

Problem 3.2.
In the US the date is written in the order MM/DD/YYYY. In Europe it's written as DD/MM/YYYY. For what fraction of the days in 2022 that we can not be determine the day from the date alone?

Problem 3.3.
There are 10 boys and 15 girls sitting around a round table in some order. Each person has 2 neighbors: one on the left and one on the right. Suppose that there are 7 distinct boy-boy pairs of neighbors. How many girl-girl pairs are there?

Problem 3.4.
How many zeroes does $11^{100} - 1$ end with?

Problem 3.5.
Julia wrote all the strings which one can obtain from the word DIFFERENTIAL by crossing our two letters (i.e. DIFRENTIAL or IFFEENTIAL). Olivia did the same with the word EXTRAPOLATES. Who got more unique strings and by how much?

4 Probability

Problem 4.1.
You can choose exactly one of these two games to play with an instructor:

- A number from 1 to 10 is selected at random. You gain full score of this problem if the number is even, and you gain no point otherwise.
- You play two round of the following game: A number from 1 to 10 is selected at random, and you win the round if the number is divisible by 3. If you win the first round, then you gain full score of this problem. If you lose the first round but win the second round, you gain one half o the score of this problem. If you lose both rounds then you gain no points.

Problem 4.2.
Teams Red and Team Blue are playing a series of games. Team Red needs to win two games to win the competition, while Team Blue needs to win three games to win the competition. Suppose that, for each game, both teams have equal winning probability. What is the probability that Team Red wins the competition?

Problem 4.3.
A probabilist select 3 points randomly from a line of 5 metres. What is the probability that these three points are at least one meter apart from each other?

Problem 4.4.
Leo Messi shoots penalties in a soccer/football field. He misses the first shot but scores the second. After that, the probability that he scores the next shot is equal to the proportion of shots he has hit so far. What is the probability that he hits exactly 75 of his first 100 shots?

Problem 4.5.
You have coins $C_1, C_2, \ldots, C_{10}$. For each k, coin C_k has probability $\frac{1}{2k+1}$ of falling heads. If all 10 coins are tossed, what is the probability that the number of head is odd?

5 Number theory

Problem 5.1.
Find all positive integers (a, b, c) such that $a \leq b \leq c$ and the least common multiple of a, b, c is $a + b + c$.

Problem 5.2.
Find all integers solutions of $x^2 + y^2 = 3(u^2 + v^2)$. Hint: $mod\, 3$ on both sides and use the method of infinite descent.

Proof. The only integer solution is $x = y = u = v = 0$. Using the fact that if 3 divides $x^2 + y^2$ then 3 divides both x and y; we then do the method of infinite descent. □

Problem 5.3.
If p, q are prime numbers and $x^4 - px^3 + q = 0$ has an integer solution. Find p, q.

Problem 5.4.
Find all positive integers (n, k) such that $(n + 1)^k - 1 = n!$.
Note that $n! = n \times (n - 1) \times \ldots \times 2 \times 1$ is the product of the first n positive integers.

Problem 5.5.
Find the smallest positive integer k that can be written as $19^a - 5^b$, where a, b are positive integers. Hint: the correct answer is the one that is the easiest to guess. You can try $mod\, 2$ which gives that the smallest such integer is even. You can try mod out some other primes from 2 to 19.