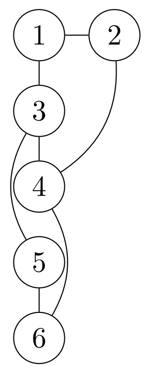
ADVANCED 2 COMPETITION II WINTER 2022

OLGA RADKO MATH CIRCLE
ADVANCED 2
MARCH 13, 2022

1. Trial 1

Problem 1.1 (2 points)**.** Find the number of spanning trees of the following graph.



Problem 1.2 (2 points)**.** Let ABC be a triangle such that point B lies inside the circle with diameter AC. What is the smallest possible value for $\angle ABC$?

Problem 1.3 (2 points)**.** A wolf, a rabbit, a turnip, and a shepherd are on one side of a river. The shepherd has a small raft and can carry at most one of the three to the other side with him. However, if left alone without the shepherd, the wolf will eat the rabbit, and the rabbit will eat the turnip. What is the minimum number of times the shepherd must cross the river in order to move all three to the other side safely? (Crossing and coming back counts as two times.)

Problem 1.4 (2 points)**.** Teams Red and Team Blue are playing a series of games. Team Red needs to win two games to win the competition, while Team Blue needs to win three games to win the competition. Suppose that, for each game, both teams have equal winning probability. What is the probability that Team Red wins the competition?

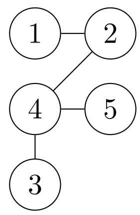
Problem 1.5 (2 points)**.** How many positive integers n between 1 and 100 (i.e. $1 \leq n \leq 100$) such that $\sqrt{n}$ is a rational number?

2. Trial 2

Problem 2.1 (2 points)**.** Find all integers n such that $\frac{n^2+3}{n-1}$ is an integer.

Problem 2.2 (2 points)**.** Recall the following definition of the Prufer code of a tree T: Start with an empty list. Then, keep removing the leaf with the smallest label until two vertices remain. Each time a leaf is removed, add its neighbor to the list. The resulting length $n-2$ sequence is the Prufer code of T.

Find the Prufer code of the following tree.



Problem 2.3 (2 points)**.** Find a rational coefficients polynomial of degree four such that $\sqrt{2}+\sqrt{3}$ is a root to the polynomial.

Problem 2.4 (2 points)**.** Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that EF is perpendicular to DB, and A and C lie on DE and DF, respectively. What is the length of EF?

Problem 2.5 (3 points)**.** Find the next number in the following sequence:

$$0, \quad 0, \quad 3, \quad 20, \quad 115, \quad 714, \quad 5033, \ldots.$$

3. Trial 3

Problem 3.1 (3 points)**.** Find all the rational roots of polynomial $2x^3 + x^2 - 5x + 2$.

Problem 3.2 (3 points)**.** Recall the definition of von Neumann ordinal from the Peano's axiom worksheet. Write 3 as a Von Neumann ordinal.

Problem 3.3 (3 points)**.** Leo Messi shoots penalties in a soccer/football field. He misses the first shot but scores the second. After that, the probability that he scores the next shot is equal to the proportion of shots he has hit so far. What is the probability that he hits exactly 75 of his first 100 shots?

Problem 3.4 (3 points)**.** The diagonals of convex quadrilateral $ABCD$ meet at point P. The area of the triangles ABP, BCP, CDP is equal to $2, 3, 4$ respectively. Find the area of triangle ADP.

Problem 3.5 (4 points)**.** There are three vertical pegs, and on one of them, there are 5 disks of increasing size (the smallest is on top). The goal is to move them all onto either of the other pegs. However, you may only move them one at a time, and no disk may ever be on top of a disk smaller than itself. What is the minimum number of moves needed to move all the disks?

4. Trial 4

Problem 4.1 (3 points)**.** You can choose exactly one of these two games to play with an instructor:

- A number from 1 to 10 is selected at random. You gain full score of this problem if the number is even, and you gain no point otherwise.
- You play two round of the following game: A number from 1 to 10 is selected at random, and you win the round if the number is divisible by 3. If you win the first round, then you gain full score of this problem. If you lose the first round but win the second round, you gain one half o the score of this problem. If you lose both rounds then you gain no points.

Problem 4.2 (3 points)**.** Let $f(x)$ be an integer coefficients polynomial $f(x)$ such that $1+\sqrt{2}$ is a root. Find another irrational number that is always a root of $f(x)$.

Problem 4.3 (3 points)**.** How many ways are there to arrange 5 open and 5 close parenthesis (the characters $'('$ and $')'$ respectively) so that the result is balanced? A balanced parenthetical expression is one where every open parenthesis is matched to exactly one unique close parenthesis: for example, "()" is balanced, while "(()", ")(", and ")" are not.

Problem 4.4 (3 points)**.** You have coins $C_1, C_2, \ldots, C_{10}$. For each k, coin C_k has probability $\frac{1}{2k+1}$ of falling heads. If all 10 coins are tossed, what is the probability that the number of head is odd?

Problem 4.5 (4 points)**.** A point E is taken on side AC of the triangle ABC. Through E pass straight lines DE and EF parallel to sides BC and AB, respectively; where D and E are points on AB and BC, respectively. Suppose that the area of the triangle ADE is 4, and the area of the triangle EFC is 3. Find the area of the quadrilateral $BDEF$.

5. Trial 5

Problem 5.1 (4 points)**.** There are 12 coins, 11 of which are identical, and the last one is visually identical but a different weight. (It could be heavier or lighter, you don't know.) Given a two-sided scale (you can weigh groups of coins together), give a method to identify the different coin after just 3 uses of the scale.

Problem 5.2 (4 points)**.** Let ABC be a triangle, and let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Suppose that we have $3\alpha + 2\beta = 180°$, and that both AC and BC have length 1. Compute the length of AB.

Problem 5.3 (4 points)**.** Determine the value of

$$\sum_{n=0}^{\infty} 4\frac{(-1)^n}{2n+1}.$$

Problem 5.4 (5 points)**.** Two real numbers, x and y, are chosen at random (uniformly) in the interval $[0,1]$. What is the probability that the closest integer to $\frac{x}{y}$ is even? Hint: Problem 5.3 will be helpful.

Problem 5.5 (5 points)**.** Let S be the smallest set of positive integers such that

(a) 2 is in S,
(b) n is in S whenever n^2 is in S, and
(c) $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S?