Math Circle Advanced 1

Winter quarter 2022 Final game

Problem 0.1.
How many zeroes are at the end of the number 2022! ?

Problem 0.2.
Find $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}$.

Problem 0.3.
Find all integers n such that $\frac{n^2+3}{n-1}$ is an integer.

Problem 0.4.
If p, q are prime numbers and $x^4 - px^3 + q = 0$ has an integer solution. Find p, q.

Problem 0.5.
A point E is taken on side AC of the triangle ABC. Through E pass straight lines DE and EF parallel to sides BC and AB, respectively; where D and E are points on AB and BC, respectively. Suppose that the area of the triangle ADE is 4, and the area of the triangle EFC is 3. Find the area of the quadrilateral $BDEF$.

Problem 0.6.
The diagonals of convex quadrilateral $ABCD$ meet at point P. The area of the triangles ABP, BCP, CDP is equal to $2, 3, 4$ respectively. Find the area of triangle ADP.

Problem 0.7.
Leo Messi shoots penalties in a soccer/football field. He misses the first shot but scores the second. After that, the probability that he scores the next shot is equal to the proportion of shots he has hit so far. What is the probability that he hits exactly 75 of his first 100 shots?

Problem 0.8.
Let ABC be a triangle, and let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Suppose that we have $3\alpha + 2\beta = 180°$, and that both AC and BC have length 1. Compute the length of AB.

Problem 0.9.
Write 4 algebraic numbers that are not rational numbers, *and explain why they are algebraic and irrational.

Problem 0.10.
Find a polynomial with integer coefficients that has $\sqrt{5} - \sqrt[3]{2}$ as a root.

Problem 0.11.
Find all positive integers (a, b, c) such that $a \leq b \leq c$ and the least common multiple of a, b, c is $a + b + c$.

Problem 0.12.
Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that EF is perpendicular to DB, and A and C lie on DE and DF, respectively. What is the length of EF?

Problem 0.13.
A country has 100 cities and some roads connecting them. Every road connects 2 cities and every city has 5 roads going from it. What is the total number of roads?

Problem 0.14.
There are 10 boys and 15 girls sitting around a round table in some order. Each person has 2 neighbors: one on the left and one on the right. Suppose that there are 7 distinct boy-boy pairs of neighbors. How many girl-girl pairs are there?

Problem 0.15.
Let $P(n)$ be a polynomial of degree 3 such that $P(0) = 1$, $P(1) = 4$, $P(2) = 11$ and $P(3) = 28$. What is $P(4)$?

Problem 0.16.
Julia wrote all the strings which one can obtain from the word DIFFERENTIAL by crossing our two letters (i.e. DIFRENTIAL or IFFEENTIAL). Olivia did the same with the word EXTRAPOLATES. Who got more unique strings and by how much?

Problem 0.17.
Teams Red and Team Blue are playing a series of games. Team Red needs to win two games to win the competition, while Team Blue needs to win three games to win the competition. Suppose that, for each game, both teams have equal winning probability. What is the probability that Team Red wins the competition?

Problem 0.18.
In the US the date is written in the order MM/DD/YYYY. In Europe it's written as DD/MM/YYYY. For what fraction of the days in 2022 that we can not be determine the day from the date alone?

Problem 0.19.
A probabilist select 3 points randomly from a line of 5 metres. What is the probability that these three points are at least one meter apart from each other?

Problem 0.20.
Given a 6x6 square grid, we say two rectangles are different on the grid if they don't contain the EXACT same squares. How many possible rectangles can be drawn using the individual squares of the grid?

Problem 0.21.
Find the smallest positive integer k that can be written as $19^a - 5^b$, where a, b are positive integers. Hint: the correct answer is the one that is the easiest to guess. You can try $mod\, 2$ which gives that the smallest such integer is even. You can try mod out some other primes from 2 to 19.

Problem 0.22.
Let ABC be a triangle such that point B lies inside the circle with diameter AC. What is the smallest possible value for $\angle ABC$?

Problem 0.23.
You have coins $C_1, C_2, \ldots, C_{10}$. For each k, coin C_k has probability $\frac{1}{2k+1}$ of falling heads. If all 10 coins are tossed, what is the probability that the number of head is odd?

Problem 0.24.
How many zeroes does $11^{100} - 1$ end with?

Problem 0.25.
Find all integers solutions of $x^2 + y^2 = 3(u^2 + v^2)$. Hint: $mod\,3$ on both sides and use the method of infinite descent.

Proof. The only integer solution is $x = y = u = v = 0$. Using the fact that if 3 divides $x^2 + y^2$ then 3 divides both x and y; we then do the method of infinite descent. □

Problem 0.26.
Find all positive integers (n, k) such that $(n + 1)^k - 1 = n!$.
Note that $n! = n \times (n - 1) \times \ldots \times 2 \times 1$ is the product of the first n positive integers.