Olga Radko Math Circle Competition

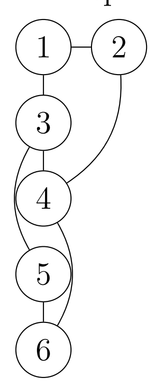
Advanced 3

Week 10 Winter 2022

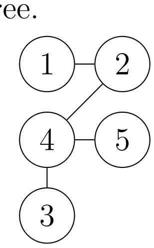
Unless otherwise noted problems may be attempted as many times as you want without any penalty; however this may be changed at the discretion of the instructor.
Problems with a numerical solution will usually be given full points for just the correct numerical value; however this is up to the discretion of the instructor (especially for problems with easily guessed numerical answers).

1. Spanning Trees

Problem 1.1 (2 points)**.** Find the number of spanning trees of the following graph.



Problem 1.2 (2 points)**.** Recall the following definition of the Prufer code of a tree T: Start with an empty list. Then, keep removing the leaf with the smallest label until two vertices remain. Each time a leaf is removed, add its neighbor to the list. The resulting length $n-2$ sequence is the Prufer code of T.
Find the Prufer code of the following tree.



2. Geometry

Hint: It will be a good idea to draw these problems when solving them.

Problem 2.1 (2 points)**.** Let ABC be a triangle such that point B lies inside the circle with diameter AC. What is the smallest possible value for $\angle ABC$?

Problem 2.2 (2 points)**.** The diagonals of convex quadrilateral $ABCD$ meet at point P. The area of the triangles ABP, BCP, CDP is equal to $2, 3, 4$ respectively. Find the area of triangle ADP.

Problem 2.3 (2 points)**.** Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that EF is perpendicular to DB, and A and C lie on DE and DF, respectively. What is the length of EF?

Problem 2.4 (2 points)**.** Let ABC be a triangle, and let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Suppose that we have $3\alpha + 2\beta = 180°$, and that both AC and BC have length 1. Compute the length of AB.

Problem 2.5 (2 points)**.** A point E is taken on side AC of the triangle ABC. Through E pass straight lines DE and EF parallel to sides BC and AB, respectively; where D and E are points on AB and BC, respectively. Suppose that the area of the triangle ADE is 4, and the area of the triangle EFC is 3. Find the area of the quadrilateral $BDEF$.

3. Probability

Problem 3.1 (4 points)**.** You can choose exactly one of these two games to play with an instructor:

- A number from 1 to 10 is selected at random. You gain full score of this problem if the number is even, and you gain no point otherwise.
- You play two round of the following game: A number from 1 to 10 is selected at random, and you win the round if the number is divisible by 3. If you win the first round, then you gain full score of this problem. If you lose the first round but win the second round, you gain one half o the score of this problem. If you lose both rounds then you gain no points.

Problem 3.2 (2 points)**.** Teams Red and Team Blue are playing a series of games. Team Red needs to win two games to win the competition, while Team Blue needs to win three games to win the competition. Suppose that, for each game, both teams have equal winning probability. What is the probability that Team Red wins the competition?

Problem 3.3 (2 points)**.** A probabilist select 3 points randomly from a line of 5 metres. What is the probability that these three points are at least one metre apart from each other?

Problem 3.4 (2 points)**.** Leo Messi shoots penalties in a soccer/football field. He misses the first shot but scores the second. After that, the probability that he scores the next shot is equal to the proportion of shots he has hit so far. What is the probability that he hits exactly 75 of his first 100 shots?

Problem 3.5 (3 points)**.** Suppose that you roll a three sided die. What is the expected time to see 1231?

Problem 3.6 (3 points)**.** You have coins $C_1, C_2, \ldots, C_{10}$. For each k, coin C_k has probability $\frac{1}{2k+1}$ of falling heads. If all 10 coins are tossed, what is the probability that the number of head is odd?

Problem 3.7 (4 points)**.** Bob is playing with quantum particles. There are four types of particles available, protons, neutrons, anti-protons, and anti-neutrons. He has a long tube (with diameter enough to fit exactly one particle) that is initially empty. Each second he picks a particle at random and shoves it down the tube. Of course if a particle and its anti-particle come into contact they annihilate. What is the probability that the tube will be empty at some time after putting in the first particle? Hint: the answer is not 1.

Problem 3.8 (4 points)**.** Two real numbers, x and y, are chosen at random (uniformly) in the interval $[0, 1]$. What is the probability that the closest integer to $\frac{x}{y}$ is even?

4. Puzzles

Problem 4.1 (4 points)**.** Devise a strategy for the following game, and then gather 7 people to play it with Aaron. You may play twice, and each time if you succeed you get 2 points.
The game: Each person is secretly assigned a hat color, either black or white. Each person sees everyone else's hat, but not their own. Then each person guesses their hat color, in a private message to Aaron, or passes. You win (getting two points) if at least one person guesses, and no one guesses incorrectly.
The hat colors will be assigned at random, beforehand, and the same colors will be given to each team.

Problem 4.2 (2 points)**.** A wolf, a rabbit, a turnip, and a shepherd are on one side of a river. The shepherd has a small raft and can carry at most one of the three to the other side with him. However, if left alone without the shepherd, the wolf will eat the rabbit, and the rabbit will eat the turnip. What is the minimum number of times the shepherd must cross the river in order to move all three to the other side safely? (Crossing and coming back counts as two times.)

Problem 4.3 (3 points)**.** Find the next number in the following sequence:

$$0, \quad 0, \quad 3, \quad 20, \quad 115, \quad 714, \quad 5033, \ldots.$$

Problem 4.4 (3 points)**.** There are three vertical pegs, and on one of them, there are 5 disks of increasing size (the smallest is on top). The goal is to move them all onto either of the other pegs. However, you may only move them one at a time, and no disk may ever be on top of a disk smaller than itself. What is the minimum number of moves needed to move all the disks?

5. General Problems

Problem 5.1 (2 points)**.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

Problem 5.2 (4 points)**.** Let S be the smallest set of positive integers such that

(a) 2 is in S,
(b) n is in S whenever n^2 is in S, and
(c) $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S?

Problem 5.3 (4 points)**.** Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$