# ORMC Intermediate 2 Winter Competition

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Welcome to the last week of this quarter's math circle! The tournament will consist of 17 questions, and your team will be given two of them at a time and will not be given any additional ones until you correctly solve or give up on one of them. The questions are a mix of computational questions and short proofs. You will be awarded 1 point for solving a problem first try, 0.5 points second try, and 0 points if you exhaust both tries and/or give up on a problem. The final problem is worth twice the points.

**Problem 1.** An animal with $n$ cells is a connected figure consisting of $n$ equal-sized cells. A dinosaur is an animal with at least 2007 cells. It is said to be primitive it its cells cannot be partitioned into two or more dinosaurs. Find the maximum number of cells in a primitive dinosaur.

**Problem 2.** One-inch squares are cut from each corner of a five-inch square. What is the area in square inches of the largest square that can be fitted into the remaining space?

**Problem 3.** The digits $1, 2, 3, 4, 5$ are each used once to write a five-digit number $PQRST$. The three-digit numbers $PQR$, $QRS$, and $RST$, are divisible by $4, 5$, and $3$, respectively. Find $P$.

**Problem 4.** Which of the following polynomials are irreducible over $\mathbb{Q}$? $p(x) = x^3 - 5$, $q(x) = x^4 - x^3 + 2x^2 - x - 1$, $r(x) = 50x^5 - 150x^2 + 42$, $s(x) = x^6 + 2x^5 + 4x^4 + 8x^3 + 16x^2 + 32x + 66$

**Problem 5.** Find the greatest common divisor of 4314464 and 2034747.

**Problem 6.** Let $(G, *)$ be a cyclic group of $n$ elements, where $n \in \mathbb{N}$. Let $x$ be an element of $G$. Show that $x^n = e$.

**Problem 7.** Let $(G, *)$ be a cyclic group of $n$ elements, where $n \in \mathbb{N}$. Let $x$ be an element of $G$. Show that the order of $x$ divides $n$.

**Problem 8.** Let $n$ be a positive integer. A test has $n$ problems, and was taken by several students. Exactly three students solved each problem, each pair of problems has exactly one student that solved both and no student solved every problem. Find the maximum possible value of $n$.

**Problem 9.** One day the candy store sold 252 sodas to 100 different customers (each of whom bought at least one soda). What is the maximum possible median number of sodas bought by customers on that day?

**Problem 10.** 10 divers are lined up in 3 lines of 2, 3, and 5 people each. Every minute, one of the divers at the front of a line jumps into the water. In how many orders can the divers jump into the water?

**Problem 11.** Similarly to how 2D shapes have symmetries, 3D shapes also have symmetric groups, consisting of rotations (now in 3D and possibly across multiple axes) and inversions (this time across a plane rather than a line). What is the order of the tetrahedral group and is the group Abelian?

**Problem 12.** You are given three piles of identical red, blue and green balls, where each pile contains at least 10 balls. In how many ways can 10 balls be selected if exactly one red ball and at least one blue ball must be selected?

**Problem 13.** A superpermutation is a string that contains all possible permutations of some characters as a substring. For instance, for the characters a and b, **aba** is a valid superpermutation as it contains every possible permutation of a and b, that is it contains **ab**a and a**ba**.

Find the shortest possible superpermutation of the characters a, b, and c.

**Problem 14.** Take the group $(\mathbb{Z}/2022\mathbb{Z}, *)$. What is the group inverse of 107 (that is, find $x$ such that $107x \mod 2022 = 1$. (Hint: Investigate the Extended Euclidean algorithm)

**Problem 15.** Squares $ABCD, EFGH, GHIJ$ are all $1 \times 1$ squares, such that points $C$ and $D$ are the midpoints of sides $IH$ and $HE$, respectively. What is the area of pentagon $AJICB$?

**Problem 16.** Let $R$ be a set of nine distinct integers. Six of its elements are $2, 3, 4, 6, 9, 14$. How many possible medians does $R$ have?

**Problem 17.** (Final Problem) A calculator is programmed to calculate a fixed polynomial $P$ with positive integer coefficients and unknown degree. When you input a positive integer $n$, it will display the value $P(n)$, which counts as one step. You are allowed to repeat this process. Try to figure out the polynomial $P$ in as few steps as possible.

When you submit your answer, the instructor will have two polynomials prepared in advance. You will be asked to guess the polynomial after at most 5 steps (for practical concerns), and if correct you will get $(6 - n)/4$ (where $n$ is the number of guesses) or 1 point, whichever is lower, for each polynomial, for a total of up to two points.