# IRRATIONAL NUMBERS

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ADVANCED 2
FEBRUARY 28, 2022

## 1. Rational Numbers

Recall that rational numbers can be viewed as fractions of the form $\frac{a}{b}$, where $a$ and $b$ are relatively prime integers and $b$ is nonzero. Here that $a$ and $b$ are relatively prime integers means $a$ and $b$ has no common divisor except $\pm 1$.

One way to view real numbers is using decimal digits representation. That is, we can treat each real number as a decimal number with infinite decimal digits.

Of course, we need add two more observations here:

(1) for real numbers like 0.2, we treat it as $0.20000\cdots$ with infinitely many zero.
(2) the representation is not unique. 1, for instance, can be written as $1.000\cdots$ and $0.999\cdots$.

Notice that, for instance, $0.999\cdots$ and $1.6142857142857\cdots$ are numbers with recursive patterns in their infinite decimal digits representations.

**Problem 1.1.** Write $0.3333...\cdots$ as a fraction.

**Solution.** $0.\overline{3} = \frac{1}{3}$

**Problem 1.2.** Write $0.123123123\cdots$ as a fraction.

**Solution.** $0.\overline{123} = \frac{123}{999} = \frac{41}{333}$

**Problem 1.3.** Write $1.6142857142857\cdots$ as a fraction.

**Solution.** $1.6 + 0.0\overline{142857} = \frac{8}{5} + \frac{1}{70} = \frac{113}{70}$

**Problem 1.4.** (Optional) Prove that, under this point of view, rational numbers has representations that either has finitely many nonzero decimal digits or has a recursive pattern in their infinite decimal digits. You may find the following steps helpful.

- Given $r = \frac{a}{b}$ with $b$ has only prime factors of 2 or 5, then $r$ has finitely many nonzero decimal digits.
- If $a$ and $b$ are two coprime integers, then there is a positive integer $n$ such that $b$ divides $a^n - 1$. This is known as Euler's Theorem.
- Conclude that if $s = \frac{a}{b}$ with $b$ has prime factors other than 2 or 5, then $s$ has a infinite decimal digits with certain recursive pattern.

**Solution.** The key observation is that an integer $b$, which is coprime with 2 and 5, is a divisor of an integer of the form $99\cdots99$. To prove this, we use the Euler's Theorem.

Notice that if $b$ is an integer that is coprime with 2 and 5, it is then coprime with 10. By Euler's theorem, $b$ divides $10^n - 1$ for some positive integer $n$. This ensures that $b$ is a divisor of $99\cdots99$ with $n$ 9's.

Now it is not hard to see that if a rational number has finite decimal representation, then its denominator is a divisor or $10^n$ for some n. If the representation has an infinite recursive pattern, then it is the sum of possible two fractions, one with denominator that only has prime factor 2 or 5, the other with denominator that is a divisor of $99\cdots99$ with $n$ 9's, for some integer $n$.

A rational number has decimal representation that either has finitely many nonzero decimal digits or has a recursive pattern in their infinite decimal digits. However, not all real numbers can be written in this form. Those real numbers that cannot be written in this form are called irrational numbers. $\sqrt{2}$, for instance, is a famous irrational number which led to The First Mathematical Crisis. The Crisis ended with the extension of field of rational numbers to field of real numbers.

## 2. Irrational Numbers

The Pythagorean school of mathematics originally believed that only rational numbers exist. However, they found that, by the Pythagorean Theorem, the hypotenuse of a isosceles right triangle with sides length 1 is a number that could not be expressed as a rational number.

Below is a short fable found online:

The secret of irrational numbers was carefully kept by the Pythagoreans. The reason for this is that the secret created a sort of crisis in the very roots of Pythagorean beliefs. There is an interesting account (its historical accuracy is not certain) about one member of the Pythagorean circle who apparently divulged the secret to someone outside the brotherhood. The traitor was thrown into deep waters and drowned. This episode is sometimes referred to as the first martyr of science. However, we could also think about this person as one of the many martyrs of superstition, since it was not the scientific aspect of irrational numbers that was the root cause of this homicide, but rather its religious extrapolations that were seen as a threat to the foundation of Pythagorean mysticism.

Today we express that the hypotenuse of a isosceles right triangle with sides length 1 as $\sqrt{2}$. That is, $\sqrt{2}$ is a number such that the square of this number is equal to 2.

**Problem 2.1.** Show that $\sqrt{2}$ is a irrational number.

**Solution.** Prove by contradiction or method of infinite descent. For instance, if we assume that $\sqrt{2} = \frac{r}{s}$, then $r^2 = 2s^2$. We then get $r$ is even. If we further write $r = 2k$, then it implies that $s^2 = 2k^2$, which leads to that $s$ is even.

More generally, we have

**Problem 2.2.** Show that $\sqrt{p}$ is an irrational number for any prime number $p$.

**Solution.** Prove by contradiction or method of infinite descent.

**Problem 2.3.** It is not hard to see the following properties that can help us when showing a real number is irrational.

- Adding a irrational number to a rational number gives a irrational number.
- Multiplying a irrational number with a nonzero rational number gives a irrational number.

**Solution.** Prove by contradiction.

Level-up time!

**Problem 2.4.** Show that $\sqrt{3+2\sqrt{2}}$ is a irrational number.

**Solution.** $\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$

**Problem 2.5.** Show that $\sqrt{7+3\sqrt{11}}$ is a irrational number.

**Solution.** Prove by contradiction.

Before we show a few more numbers that are irrational, we need a powerful theorem that will help us later. The following theorem is well-known as the Rational Root Theorem.

**Problem 2.6.** For a polynomial with integer coefficients, $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, if $\frac{r}{s}$, with $r$ and $s$ relatively prime, is a rational root to the polynomial, then $r$ is a divisor of $a_0$ and $s$ is a divisor of $a_n$.

**Solution.** Plug in $x = \frac{r}{s}$ into the polynomial and clear the denominator by multiplying both sides of the equation with $s^n$.

**Problem 2.7.** Prove that $\sqrt[n]{p}$ is a irrational number for any prime number $p$ and any positive integer $n \geq 2$.

**Solution.** $x^n - p = 0$ only has possible rational roots $\pm p$. Plug in and check neither is a root.

**Problem 2.8.** Use the Rational Root Theorem to prove that $\sqrt[3]{3+2\sqrt{2}}$ is a irrational number.

**Solution.** Write $x = \sqrt[3]{3+2\sqrt{2}}$. Then we get that $x^6 - 6x^3 + 1 = 0$, which only has possible rational roots $\pm 1$. Plug in and check neither is a root.

## 3. Analysis

We have seen the power of algebra in showing a real number is irrational. Most of the work is carried out by equalities. Now it is time to see how inequalities and analysis come into play.

You might have seen $e$, the base of natural logarithm, already. There are several equivalent ways to define this important constant.

One way to define $e$ is as the following infinite sum:

$$e \ := \ 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \ + \ \ldots \ = \ \sum_{n=0}^{\infty} \frac{1}{1 \cdot 2 \cdot \cdots \cdot n},$$

It is not trivial to show that above defines a real number that is finite, but we will see it once we solve the next question.

**Problem 3.1.** Show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is not an integer. Hint: show that $2 < e < 3$.

> **Solution.** It is not hard to see that $e > 2$. Notice that $0 < \sum_3^{\infty} \frac{1}{n!} < \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \cdots \leq \frac{1}{2}$. This results that $e < 3$.

**Problem 3.2.** Show that $e = \sum_0^{\infty} \frac{1}{n!}$ is an irrational number.

> **Solution.** Prove by contradiction. Assume $e = \frac{p}{q}$, where $p$, $q$ are relatively prime. From the previous problem, we know that $q \neq 1$. Multiply both sides by $q!$. We may assume that $q \geq 2$.
>
> Notice that $0 < \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \leq \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+2)(q+3)} + \cdots \leq \frac{2}{q+1} < 1$. This results in a contradiction.

$e$ prevails in nature. For instance, for constant $a > 0$, $a^x$ is an exponential function. The derivative function of $a^x$ is itself if and only if $a = e$. This too is non-trivial to prove. Also one might have seen that $e^r = \lim_{n\to\infty}(1 + \frac{r}{n})^n$, which has applications in compound interest rate.

## 4. Set Theory

Irrational numbers are prevailing on the number line. In fact, if you choose a number randomly from $[0, 1]$, the probability that the number is irrational turns out to be 1. This is an interesting result in probability theory.

In this session, instead of measure theory, we are going to see the set of irrational numbers are uncountable, giving us the intuition that there are way more irrational numbers then rational numbers on the number line.

**Problem 4.1.** Find a bijection between $\mathbb{N}$, the set of of natural numbers, and $\mathbb{Z}$, the set of integers.

> **Solution.** We could map 0 to 0, odd nature numbers to positive integers, and nonzero even nature numbers to negative integers.

**Problem 4.2.** Find a bijection between $\mathbb{N}$, the set of of natural numbers, and $\mathbb{Q}$, the set of rational numbers.

If you are viewing this note online, you might find this link on bijection helpful. Otherwise, please take a look at the following examples if you would like a warm up with bijective maps.

- $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ is not surjective as we cannot find a preimage for -1.

- $f(x) = x^2$ from $\mathbb{R}$ to $[0, \infty)$ is not injective since $f(-1) = 1 = f(1)$.
- $f(x) = x^2$ from $[0, \infty)$ to $[0, \infty)$ is both injective and surjective, thus bijective.

**Solution.** We just need a bijection from $\mathbb{Z}$ to $\mathbb{Q}$. We could map 0 to 0 and all we need to do is making positive rational numbers enumerable, which could be done by diagonal method.

If a set is finite or can be bijectively mapped to $\mathbb{N}$, we say the set is countable. Above questions implies that $\mathbb{Z}$ and $\mathbb{Q}$ are countable. Sets that are not countable are called uncountable.

**Problem 4.3.** Prove that the set of real numbers are uncountable. You might find the following Cantor's diagonal construction helpful.

- Suppose $\mathbb{R}$ is countable. Arrange the list of real numbers as $\{x_1, x_2, x_3, \cdots\}$.
- Here we write each real number in its infinite decimal representation.
- Consider $x$ defined to be $0.a_1a_2a_3\cdots$, where for any positive integer $i$, $a_i = 7$ if the $i$-th decimal digit (after the decimal dot) of $x_i$ is not 7, and $a_i = 3$ if the $i$-th decimal digit (after the decimal dot) of $x_i$ is 7.
- This $x$ we constructed is then not in the list of real numbers we set up at the beginning. (Notice that we choose 3 and 7 instead of 0 or 1 here to avoid non-unique decimal representation of real numbers).

**Solution.** Prove by contradiction by Cantor's diagonal construction.

**Problem 4.4.** Prove that the union of two countable sets is countable.

**Solution.** Similar to the proof of showing $\mathbb{Z}$ and $\mathbb{Q}$ are bijectively same.

**Problem 4.5.** Prove that the set of irrational numbers are uncountable.

**Solution.** Prove by contradiction using the previous questions.

## 5. Bonus: Miscellaneous Problems

**Problem 5.1.** Does there exist irrational numbers $a, b$ such that $a^b$ is rational? What about an irrational number $a$ such that $a^a$ is rational?

**Solution.** Yes to both. In the first case, take $\sqrt{2}^{\sqrt{2}}$. If $\sqrt{2}^{\sqrt{2}}$ is rational, take $a = \sqrt{2}$ and $b = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is not rational, then $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ is rational. Take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

Let $x$ be a solution to $x^x = 2$, which exists by continuity/IVT. Suppose for contradiction $x^x$ is rational and proceed like the $\sqrt{2}$ proof.

**Problem 5.2.** When is $\log_a(b)$ irrational for $a$ and $b$ integers? Give a condition and proof of correctness.

**Solution.** Suppose it is rational, so $\log_a(b) = \frac{p}{q}$, so $a^{p/q} = b$, so $a^p = b^q$. Take prime factorization to see that this has a solution only when $a$ and $b$ are integer powers of the same number.

## 6. Bonus: Additional Applications of Diagonalization

Cantor's diagonalization argument for the uncountability of the reals is a common technique that can be used to show many claims.

**Problem 6.1.** Given functions $f, g : \mathbb{N} \to \mathbb{N}$, we say that $f$ dominates $g$ if $f(x) \geq g(x)$ for all $x \in \mathbb{N}$. Let $A$ be a set of functions from $\mathbb{N}$ to $\mathbb{N}$ such that for every $g : \mathbb{N} \to \mathbb{N}$, there exists $f \in A$ that dominates $g$. Show by diagonalization that $A$ must be an uncountable set.

**Solution.** Enumerate $A = \{f_0, f_1, f_2, \dots\}$ and construct $g(x) = f_x(x) + 1$.

Next, we will give a proof for the famous *halting problem* in computer science. Denote $\{0,1\}^* = \bigcup_{n=0}^{\infty} \{0,1\}^n$, the set of binary sequences of arbitrary finite length. A *decision problem* is function $f : \{0,1\}^* \to \{\mathrm{T}, \mathrm{F}\}$.

**Problem 6.2.** Show by diagonalization that the number of problems is uncountable.

**Solution.** First note that $\{0,1\}^*$ is countable because you can first enumerate all the strings of length 1, then 2, etc. Draw a table where the rows are problems and the columns are an enumeration of the inputs. Flipping the diagonal of the table gives a problem not in the list.

A *computer program* is a text file that tells the computer how to compute an output for every input in $\{0,1\}^*$. In other words, every program $\alpha$ computes a function $g_\alpha : \{0,1\}^* \to \{\mathrm{T}, \mathrm{F}, \#\}$, where # means that the program gets stuck in an infinite loop and never outputs anything. In this case, we say the program does not *halt*.

**Problem 6.3.** A text file is a finite sequence of keyboard characters. Show that the number of programs is countable.

**Solution.** Denoting the (finite) set of keyboard characters as $S$, a program is an element of $S^*$. We can hence enumerate programs by first enumerating all the programs of length 1, then of length 2, etc. because there are finitely many of each.

**Problem 6.4.** Using diagonalization, prove that there exists a problem $U$ such that no program $\alpha$ satisfies $g_\alpha(\beta) = U(\beta)$ for all $\beta \in \{0,1\}^*$.

**Solution.** Draw a table where the rows are the programs $g_\alpha$, the columns are the inputs $\beta$, and the cells are $g_\alpha(\beta)$. Define the problem $U$ by $U(\alpha) = \neg g_\alpha(\alpha)$, where $\neg$ denotes logical negation, and the logical negation of # is taken to be T. (F works too, just need to be consistent with solution to 5.5.)

The *halting problem* is the problem $H$ to determine if given a program $\alpha$ and input $\beta$, whether or not $g_\alpha(\beta) = \#$.

**Problem 6.5.** Prove that if there exists a program that can solve $H$, then you can construct a program to solve $U$ in Problem 6.4. Conclude that no program can solve $H$.

**Solution.** Solve $U(\alpha)$ by running $H$ on problem $\alpha$ and input $\alpha$. If it halts, then just run the code of $U$ on $\alpha$ and return the opposite answer. If it doesn't halt, return T. (Or F, consistent with 5.4.)

## 7. Bonus: Algebraic Numbers

A number $z \in \mathbb{C}$ is *algebraic* if there exists a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with integer coefficients $a_0, \ldots, a_n \in \mathbb{Z}$ such that $z$ is a root of $f$, i.e. $f(z) = 0$. For example, $\sqrt{2}$ is algebraic because it is a root of $f(x) = x^2 - 2$. The set of algebraic numbers is denoted $\overline{\mathbb{Q}}$, for reasons we'll discuss later.

**Problem 7.1.** Prove that every rational number is algebraic.

**Solution.** The number $\frac{p}{q}$ is a root of the polynomial $p - qx = 0$.

Numbers that are not algebraic are called *transcendental.* For example, $e$ is a transcendental number, but this is a little tricky to prove.

**Problem 7.2.** Draw a Venn diagram to show the relationship between $\mathbb{Q}, \overline{\mathbb{Q}}$, $\mathbb{R}$, and $\mathbb{C}$. Give an example number for every region in your diagram (no proof necessary).

**Solution.** The Venn diagram should have $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, and $\overline{\mathbb{Q}}$ should contain all of $\mathbb{Q}$ with parts of $\mathbb{R}$ and $\mathbb{C}$. As for examples, here is a sample: $1$ ($\mathbb{Q}$), $\sqrt{2}$ $((\overline{\mathbb{Q}}-\mathbb{Q})\cap\mathbb{R})$, $e(\mathbb{R})-\overline{\mathbb{Q}})$, $i$ $(\overline{\mathbb{Q}} - \mathbb{R})$, $ie$ $(\mathbb{C} - \overline{\mathbb{Q}})$.

**Problem 7.3.** Show that $\overline{\mathbb{Q}}$ is countable.

**Solution.** Every polynomial has finitely many roots. So if we can enumerate the polynomials, then we can enumerate the algebraic numbers. To enumerate the polynomials, we first enumerate polynomials of fixed degree $d$. We do this similar to enumerating

$\mathbb{Q}$ by snaking around. Then, we can enumerate all polynomials in a list where for the $n$th polynomial, we write $n = 2^d m$ for $d$ as large as possible and take the next degree $d$ polynomial.

(Note: countable union of countable sets makes the problem easier but requires AC so it is beyond the scope of this worksheet, so instructors should not suggest this idea.)

Algebraic numbers are cool because they are *the algebraic closure of the rationals.* This is why we write $\overline{\mathbb{Q}}$ and it means the following theorem:

**Problem 7.4.** (Challenge) Prove that $z$ is algebraic if and only if there exists a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ with *algebraic* coefficients $a_0, \ldots, a_n \in \overline{\mathbb{Q}}$ such that $z$ is a root of $f$.

Notice that the above theorem isn't true for $\mathbb{R}$! For $\mathbb{C}$, this is called the fundamental theorem of algebra. Sets like $\overline{\mathbb{Q}}$ and $\mathbb{C}$ that satisfy this theorem are called *algebraically closed.*