ORMC Olympiad Group
Winter: Week 6

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February 27, 2022

Problems

1. (2012W Junior Tubitak Camp) $p > 7$ is a prime number and $m = \frac{8^p-1}{7}$ chosen. Prove
$$2^{m-1} \equiv 1 (mod\ m)$$

2. **(TJNMO-FR 2018 - modified)** A positive integer is called *special* if there exist two positive integers $a \geq 5$ and $b \geq 5$ so that n gives remainder b when divided by a, and gives remainder $a-2$ when divided by b. Find the number of *special* numbers less than 100.

3. Assume p is a prime in a form of $p = 3k + 2$ where k is an positive integer. Assume $p | a^2 + ab + b^2$ for some integers a, b. Prove $p|a$.

4. (2013W Junior Tubitak Camp - Bahattin Yildiz) Find all prime pairs (p, q) such that
$$(p + q)^p = (q - p)^{2q-1}$$

5. (Ireland 2009) Find all positive integers n for which $n^8 + n + 1$ is a prime number.

6. Find the number of solution pairs (x, y) where $x, y \in \{1, 2, \ldots, 20\}$ such

that

$$\begin{aligned} 3x - y &\equiv 0 \mod 5 \\ xy &\equiv 2 \mod 5 \end{aligned}$$

7. **(AIME 1983)** Let $a_n = 6^n + 8^n$. Determine the remainder upon dividing a_{83} by 49.

8. (AIME 1988) Find the smallest positive integer whose cube ends in 888. **HINT: Say** $N^3 \equiv 888 \mod 1000$**. Writing** $1000 = 125 \cdot 8$**, we have** $N^3 \equiv 888 \mod 125$ **and** $N^3 \equiv 888 \equiv 0 \mod 8$ **(which means** N **is even). Progressively look** N **in mod 5, mod 25 and mod 125. You may start by writing** $N = 5k + r$**, and find** r **and** k **in mod 5**

9. (AIME 1988-modified) Find the smallest positive integer whose cube ends in 207.

10. **(TNMO-FR 2009-modified)** Find the largest positive integer which cannot be written as a sum of two positive composite numbers, one divisible by 7 and the other divisible by 13.

11. **(AIME 2021 II)** Find the least positive integer n for which $2^n + 5^n - n$ is a multiple of 1000.