Problems

1. Find $2^{1000^{1000}}$ in mod 7.
2. Find $2^{1000^{1000}}$ in mod 15.
3. Find the last three digits of the number $2013^{1000}$.
4. Find the last three digits of the number $2017^{2017}$.
5. (Canada MO 2003) Find the last three digits of the number $2003^{2002^{2001}}$.
6. (JBMO-2007) Let $p$ be a prime number. Show that $7p + 3^p - 4$ is not a perfect square.
7. A number $m$ in mod $n$ is called perfect square if there exist an integer $x$ such that $m \equiv x^2 \mod n$. For example, 0, 1, 4 are perfect squares in mod 5, and 0, 1, 2, 4 are perfect squares in mod 7. Find the total number of perfect squares in mod 25.
8. For which $n$, there exist complete residue class $a_0, a_1, \ldots, a_{n-1}$ in mod $n$ so that $a_0, a_1 + 1, \ldots, a_{n-1} + n - 1$ is also a complete residue class?
9. For which $n$, there exist complete residue class $a_0, a_1, \ldots, a_{n-1}$ in mod $n$ so that $a_0, a_1 + 3, a_2 + 6, \ldots, a_{n-1} + 3(n - 1)$ is also a complete residue
10. Let \( p > 2 \) be a prime number. Is there a complete residue class \( \{a_1, \ldots, a_{p-1}\} = \{1, 2, 3, \ldots, p-1\} \mod p \) so that \( a_1, 2a_2, 3a_3, \ldots, (p-1)a_{p-1} \) is also a complete residue class?

**HINT:** Consider the product and use Wilson’s theorem.

11. **(TJNMO-FR 2018 - modified)** A positive integer is called *special* if there exist two positive integers \( a \geq 5 \) and \( b \geq 5 \) so that \( n \) gives remainder \( b \) when divided by \( a \), and gives remainder \( a - 2 \) when divided by \( b \). Find the number of *special* numbers less than 100.

12. (a) **(PSS ch6-modified)** Can the number A consisting of 500 sixes and some zeros be a square?

(b) **(PSS ch6)** Can the number A consisting of 600 sixes and some zeros be a square?

13. For any integer \( n > 101 \), we define number \( M_n \) as \( M_n = 101102103 \ldots n \).

For example, \( M_{103} = 101102103 \), \( M_{251} = 101102103 \ldots 250251 \). Find the largest positive integer \( k \) such that \( 3^k | M_{400} \).

14. **(PSS 6.15)** Find all primes \( p, q \), so that \( p^2 - 2q^2 = 1 \).

15. **(PSS 6.10)** Prove if \( 2^n - 1 \) is prime, then so is \( n \).

16. Prove if \( 2^n + 1 \) is prime, then \( n \) is power of 2.

17. Primes in the form \( 2^{2^n} + 1 \) are called *Fermat’s Primes*, with the notation \( F_n = 2^{2^n} + 1 \). What are first few *Fermat Primes*. Prove or disprove: \( F_n \) as always prime.

Here is some further reading:


18. **(2012W Junior Tubitak Camp)** Solve \( x^y + 1 = z \) in prime numbers. This means that \( (x, y, z) \) are all prime numbers.

19. Assume \( p \) is a prime in a form of \( p = 3k + 2 \) where \( k \) is an positive integer. Assume \( p | a^2 + ab + b^2 \) for some integers \( a, b \). Prove \( p | a \).