ORMC Olympiad Group
Winter: Week 5
Remaining Problems

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Problems

1. Find $2^{1000^{1000}}$ in mod 7.
2. Find $2^{1000^{1000}}$ in mod 15.
3. Find the last three digits of the number 2013^{1000}.
4. Find the last three digits of the number 2017^{2017}.
5. (Canada MO 2003) Find the last three digits of the number $2003^{2002^{2001}}$.
6. **(JBMO-2007)** Let p be a prime number. Show that $7p + 3^p - 4$ is not a perfect square.
7. A number m in *mod* n is called perfect square if there exist an integer x such that $m \equiv x^2 \mod n$. For example, $0, 1, 4$ are perfect squares in *mod* 5, and $0, 1, 2, 4$ are perfect squares in *mod* 7. Find the total number of perfect squares in mod 25.
8. For which n, there exist complete residue class $a_0, a_1, \ldots, a_{n-1}$ in $\mod n$ so that $a_0, a_1 + 1, \ldots, a_{n-1} + n - 1$ is also a complete residue class?
9. For which n, there exist complete residue class $a_0, a_1, \ldots, a_{n-1}$ in $\mod n$ so that $a_0, a_1 + 3, a_2 + 6, \ldots, a_{n-1} + 3(n-1)$ is also a complete residue

class?

10. Let $p > 2$ be a prime number. Is there a complete residue class $\{a_1, \ldots, a_{p-1}\} = \{1, 2, 3, \ldots, p-1\}$ in $\mod p$ so that $a_1, 2a_2, 3a_3, \ldots, (p-1)a_{p-1}$ is also a complete residue class?
HINT: Consider the product and use Wilson's theorem

11. **(TJNMO-FR 2018 - modified)** A positive integer is called *special* if there exist two positive integers $a \geq 5$ and $b \geq 5$ so that n gives remainder b when divided by a, and gives remainder $a-2$ when divided by b. Find the number of *special* numbers less than 100.

12. (a) **(PSS ch6-modified)** Can the number A consisting of 500 sixes and some zeros be a square?

    (b) **(PSS ch6)** Can the number A consisting of 600 sixes and some zeros be a square?

13. For any integer $n > 101$, we define number M_n as $M_n = \overline{101102103\ldots n}$. For example, $M_{103} = 101102103$, $M_{251} = 101102103\ldots250251$. Find the largest positive integer k such that $3^k | M_{400}$.

14. **(PSS 6.15)** Find all primes p, q, so that $p^2 - 2q^2 = 1$.

15. (PSS 6.10) Prove if $2^n - 1$ is prime, then so is n.

16. Prove if $2^n + 1$ is prime, then n is power of 2.

17. Primes in the form $2^{2^n}+1$ are called *Fermat's Primes*, with the notation $F_n = 2^{2^n}+1$. What are first few *Fermat Primes*. Prove or disprove: F_n as always prime.

    Here is some further reading:

    $http://www.math.ualberta.ca/\ isaac/math324/s12/fermat\_numbers.pdf$

18. (2012W Junior Tubitak Camp) Solve $x^y + 1 = z$ in prime numbers. This means that (x, y, z) are all prime numbers.

19. Assume p is a prime in a form of $p = 3k + 2$ where k is an positive integer. Assume $p | a^2 + ab + b^2$ for some integers a, b. Prove $p | a$.