PROPOSITIONAL LOGIC II

MATH CIRCLE (INTERMEDIATE) 3/18/2012

Def: We say that a truth assignment satisfies a wff θ if the truth value of θ is T; in this case θ is called satisfiable. Similarly a truth assignment that satisfies a set of wffsΣ if there is a *single* truth assignment that satisfies every θ in Σ; in this case we call Σ satisfiable.

1) Are the following wffs or sets of wffs satisfiable? If so, find a truth assignment that satisfies it.

a) $\neg(P \to Q) \wedge \neg R$

b) $(P \to Q) \leftrightarrow (\neg Q \to P)$

c) $(P \wedge \neg R) \vee \neg Q$

d) $(P \vee Q) \leftrightarrow R$

e) $\Sigma = \{\neg(P \to Q) \land \neg R, (P \land \neg R) \lor \neg Q\}$

f) $\Sigma = \{\neg(P \to Q) \land \neg R, (P \lor Q) \leftrightarrow R\}$

g) $\Sigma = \{(P \land \neg R) \lor \neg Q, (P \lor Q) \leftrightarrow R\}$

Def: We say that a set of wffs Σ tautologically implies a wff θ (written $\Sigma \models \theta$) if and only if *every* truth assignment that satisfies Σ also satisfies θ. We say that θ is a tautology (written $\models \theta$) if *every* truth assignment satisfies θ.

2) Show that $\{P, P \to \neg Q, Q \leftrightarrow R\} \models \neg R$.

3) Of the following three formulas, which tautologically imply which? (That is, does $\{\theta_1\} \models \theta_2$, $\{\theta_2\} \models \theta_1$, etc.)

$$\theta_1 = (P \to Q),\ \theta_2 = \neg[(P \to Q) \to \neg(Q \to P)],\ \theta_3 = (\neg P \vee Q) \wedge (P \vee \neg Q)$$

4) Given the following, is the unicorn mythical? How about magical? How about horned?

- If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal.
- If the unicorn is either immortal or a mammal, then it is horned.
- The unicorn is magical if it is horned.

5) Show the following:

a) $\Sigma \cup \{\theta\} \models \lambda$ if and only if $\Sigma \models \theta \to \lambda$.

b) If either $\Sigma \models \theta$ or $\Sigma \models \lambda$, then $\Sigma \models \theta \vee \lambda$.

c) Is it true that if $\Sigma \models \theta \vee \lambda$ then either $\Sigma \models \theta$ or $\Sigma \models \lambda$?

6) Adams, Brown, and Clark are three suspects for a murder. Adams says, "I didn't do it. The victim was an old friend of Brown's. But Clark hated him." Brown says, "I didn't do it. I didn't even know the guy. Besides, I was out of town all that week." Clark says, "I didn't do it. I saw both Adams and Brown in town with the victim that day; one of them must have done it."

Assume that an innocent man has nothing to hide and tells the truth. Is one of the suspects necessarily guilty? If so, can we solve the murder?

Suppose Σ is a set of wffs. Define a deduction of λ from Σ to be a finite sequence $\theta_1, \ldots, \theta_n$ such that $\theta_n = \lambda$, and for every $k = 1, \ldots, n$, either a) θ_k is a tautology, b) θ_k is in Σ, or c) for some $i, j < k$, θ_j has the form $\theta_i \to \theta_k$ (and we say that θ_k is obtained from θ_i and θ_j from modus ponens). If there is a deduction of λ from Σ, we write $\Sigma \vdash \lambda$.

7) Show that $\{P, P \to \neg Q, Q \leftrightarrow R\} \vdash \neg R$. Note you *will* need some tautologies along the way.

Theorem: (Soundness) Suppose Σ is a set of wffs and λ is a wff. If $\Sigma \vdash \lambda$ then $\Sigma \models \lambda$. Note that this can be proven fairly easily using induction!

8) Suppose that $\Sigma = \{\theta_1, \ldots, \theta_n\}$ is a *finite* set of wffs and suppose that $\Sigma \models \lambda$. Show that $\{\theta_1 \wedge \cdots \wedge \theta_n\} \vdash \lambda$. Hint: If you pick the correct tautology, the deduction is only three lines!

Theorem: (Completeness) Suppose Σ is a set of wffs and λ is a wff. If $\Sigma \models \lambda$ then $\Sigma \vdash \lambda$.

Challenge 1) Explain why Problem 8) is not a proof of the Completeness Theorem. (There are a couple of things that go wrong.)

Challenge 2) Assuming the Completeness Theorem, prove the following form of the Compactness Theorem:

Theorem: (Compactness) Suppose Σ is a set of wffs and λ is a wff. If $\Sigma \models \lambda$, then there are finitely many wffs $\theta_{1,} \dots, \theta_n$ in Σ such that $\{\theta_{1,} \dots, \theta_n\} \models \lambda$.

Some problems are taken from:

- H. Enderton "A Mathematical Introduction to Logic"
- S. Russel, P. Norvig "Artificial Intelligence: A Modern Approach"