

IRRATIONAL NUMBERS

OLGA RADKO MATH CIRCLE

ADVANCED 2

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1. RATIONAL NUMBERS

Recall that rational numbers can be viewed as fractions of the form $\frac{a}{b}$, where a and b are relatively prime integers and b is nonzero. Here that a and b are relatively prime integers means a and b has no common divisor except ± 1 .

One way to view real numbers is using decimal digits representation. That is, we can treat each real number as a decimal number with infinite decimal digits.

Of course, we need add two more observations here:

- (1) for real numbers like 0.2, we treat it as 0.20000... with infinitely many zero.
- (2) the representation is not unique. 1, for instance, can be written as 1.000... and 0.999...

Notice that, for instance, 0.999... and 1.6142857142857... are numbers with recursive patterns in their infinite decimal digits representations.

Problem 1.1. Write 0.3333... as a fraction.

Problem 1.2. Write 0.123123123... as a fraction.

Problem 1.3. Write 1.6142857142857... as a fraction.

Problem 1.4. (Optional) Prove that, under this point of view, rational numbers has representations that either has finitely many nonzero decimal digits or has a recursive pattern in their infinite decimal digits. You may find the following steps helpful.

- Given $r = \frac{a}{b}$ with b has only prime factors of 2 or 5, then r has finitely many nonzero decimal digits.
- If a and b are two coprime integers, then there is a positive integer n such that b divides $a^n - 1$. This is known as Euler's Theorem.
- Conclude that if $s = \frac{a}{b}$ with b has prime factors other than 2 or 5, then s has a infinite decimal digits with certain recursive pattern.

A rational number has decimal representation that either has finitely many nonzero decimal digits or has a recursive pattern in their infinite decimal digits. However, not all real numbers can be written in this form. Those real numbers that cannot be written in this form are called irrational numbers. $\sqrt{2}$, for instance, is a famous irrational number which led to The First Mathematical Crisis. The Crisis ended with the extension of field of rational numbers to field of real numbers.

2. IRRATIONAL NUMBERS

The Pythagorean school of mathematics originally believed that only rational numbers exist. However, they found that, by the Pythagorean Theorem, the hypotenuse of an isosceles right triangle with sides length 1 is a number that could not be expressed as a rational number.

Below is a short fable found online:

The secret of irrational numbers was carefully kept by the Pythagoreans. The reason for this is that the secret created a sort of crisis in the very roots of Pythagorean beliefs. There is an interesting account (its historical accuracy is not certain) about one member of the Pythagorean circle who apparently divulged the secret to someone outside the brotherhood. The traitor was thrown into deep waters and drowned. This episode is sometimes referred to as the first martyr of science. However, we could also think about this person as one of the many martyrs of superstition, since it was not the scientific aspect of irrational numbers that was the root cause of this homicide, but rather its religious extrapolations that were seen as a threat to the foundation of Pythagorean mysticism.

Today we express that the hypotenuse of an isosceles right triangle with sides length 1 as $\sqrt{2}$. That is, $\sqrt{2}$ is a number such that the square of this number is equal to 2.

Problem 2.1. Show that $\sqrt{2}$ is an irrational number.

More generally, we have

Problem 2.2. Show that \sqrt{p} is an irrational number for any prime number p .

Problem 2.3. It is not hard to see the following properties that can help us when showing a real number is irrational.

- Adding an irrational number to a rational number gives an irrational number.
- Multiplying an irrational number with a nonzero rational number gives an irrational number.

Level-up time!

Problem 2.4. Show that $\sqrt{3 + 2\sqrt{2}}$ is an irrational number.

Problem 2.5. Show that $\sqrt{7 + 3\sqrt{11}}$ is an irrational number.

Before we show a few more numbers that are irrational, we need a powerful theorem that will help us later. The following theorem is well-known as the Rational Root Theorem.

Problem 2.6. For a polynomial with integer coefficients, $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, if $\frac{r}{s}$, with r and s relatively prime, is a rational root to the polynomial, then r is a divisor of a_0 and s is a divisor of a_n .

Problem 2.7. Prove that $\sqrt[n]{p}$ is an irrational number for any prime number p and any positive integer $n \geq 2$.

Problem 2.8. Use the Rational Root Theorem to prove that $\sqrt[3]{3 + 2\sqrt{2}}$ is an irrational number.

3. ANALYSIS

We have seen the power of algebra in showing a real number is irrational. Most of the work is carried out by equalities. Now it is time to see how inequalities and analysis come into play.

You might have seen e , the base of natural logarithm, already. There are several equivalent ways to define this important constant.

One way to define e is as the following infinite sum:

$$e := 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots = \sum_{n=0}^{\infty} \frac{1}{1 \cdot 2 \cdot \dots \cdot n},$$

It is not trivial to show that above defines a real number that is finite, but we will see it once we solve the next question.

Problem 3.1. Show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is not an integer. Hint: show that $2 < e < 3$.

Problem 3.2. Show that $e = \sum_0^{\infty} \frac{1}{n!}$ is a irrational number.

e is prevailing in nature. For instance, for constant $a > 0$, a^x is an exponential function. The derivative function of a^x is itself if and only if $a = e$. This too is non-trivial to prove. Also one might have seen that $e^r = \lim_{n \rightarrow \infty} (1 + \frac{r}{n})^n$, which has applications in compound interest rate.

4. SET THEORY

Irrational numbers are prevailing on the number line. In fact, if you choose a number randomly from $[0, 1]$, the probability that the number is irrational turns out to be 1. This is an interesting result in probability theory.

In this session, instead of measure theory, we are going to see the set of irrational numbers are uncountable, giving us the intuition that there are way more irrational numbers then rational numbers on the number line.

Problem 4.1. Find a bijection between \mathbb{N} , the set of of natural numbers, and \mathbb{Z} , the set of integers.

Problem 4.2. Find a bijection between \mathbb{N} , the set of of natural numbers, and \mathbb{Q} , the set of rational numbers.

If you are viewing this note online, you might find this link on bijection helpful. Otherwise, please take a look at the following examples if you would like a warm up with bijective maps.

- $f(x) = x^2$ from \mathbb{R} to \mathbb{R} is not surjective as we cannot find a preimage for -1.
- $f(x) = x^2$ from \mathbb{R} to $[0, \infty)$ is not injective since $f(-1) = 1 = f(1)$.
- $f(x) = x^2$ from $[0, \infty)$ to $[0, \infty)$ is both injective and surjective, thus bijective.

If a set is finite or can be bijectively mapped to \mathbb{N} , we say the set is countable. Above questions implies that \mathbb{Z} and \mathbb{Q} are countable. Sets that are not countable are called uncountable.

Problem 4.3. Prove that the set of real numbers are uncountable. You might find the following Cantor's diagonal construction helpful.

- Suppose \mathbb{R} is countable. Arrange the list of real numbers as $\{x_1, x_2, x_3, \dots\}$.
- Here we write each real number in its infinite decimal representation.
- Consider x defined to be $0.a_1a_2a_3 \dots$, where for any positive integer i , $a_i = 7$ if the i -th decimal digit (after the decimal dot) of x_i is not 7, and $a_i = 3$ if the i -th decimal digit (after the decimal dot) of x_i is 7.

- This x we constructed is then not in the list of real numbers we set up at the beginning. (Notice that we choose 3 and 7 instead of 0 or 1 here to avoid non-unique decimal representation of real numbers).

Problem 4.4. Prove that the union of two countable sets is countable.

Problem 4.5. Prove that the set of irrational numbers are uncountable.

5. BONUS: MISCELLANEOUS PROBLEMS

Problem 5.1. Does there exist irrational numbers a, b such that a^b is rational? What about an irrational number a such that a^a is rational?

Problem 5.2. When is $\log_a(b)$ irrational? Give a condition and proof of correctness.

6. BONUS: ADDITIONAL APPLICATIONS OF DIAGONALIZATION

Cantor's diagonalization argument for the uncountability of the reals is a common technique that can be used to show many claims.

Problem 6.1. Given functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, we say that f dominates g if $f(x) \geq g(x)$ for all $x \in \mathbb{N}$. Let A be a set of functions from \mathbb{N} to \mathbb{N} such that for every $g : \mathbb{N} \rightarrow \mathbb{N}$, there exists $f \in A$ that dominates g . Show by diagonalization that A must be an uncountable set.

Next, we will give a proof for the famous *halting problem* in computer science. Denote $\{0, 1\}^* = \bigcup_{n=0}^{\infty} \{0, 1\}^n$, the set of binary sequences of arbitrary finite length. A *decision problem* is function $f : \{0, 1\}^* \rightarrow \{T, F\}$.

Problem 6.2. Show by diagonalization that the number of problems is uncountable.

A *computer program* is a text file that tells the computer how to compute an output for every input in $\{0, 1\}^*$. In other words, every program α computes a function $g_\alpha : \{0, 1\}^* \rightarrow \{T, F, \#\}$, where $\#$ means that the program gets stuck in an infinite loop and never outputs anything. In this case, we say the program does not *halt*.

Problem 6.3. A text file is a finite sequence of keyboard characters. Show that the number of programs is countable.

Problem 6.4. Using diagonalization, prove that there exists a problem U such that no program α satisfies $g_\alpha(\beta) = U(\beta)$ for all $\beta \in \{0, 1\}^*$.

The *halting problem* is the problem H to determine if given a program α and input β , whether or not $g_\alpha(\beta) = \#$.

Problem 6.5. Prove that if there exists a program that can solve H , then you can construct a program to solve U . Conclude that no program can solve H .

7. BONUS: ALGEBRAIC NUMBERS

A number $z \in \mathbb{C}$ is *algebraic* if there exists a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with integer coefficients $a_0, \dots, a_n \in \mathbb{Z}$ such that z is a root of f , i.e. $f(z) = 0$. For example, $\sqrt{2}$ is algebraic because it is a root of $f(x) = x^2 - 2$. The set of algebraic numbers is denoted $\overline{\mathbb{Q}}$, for reasons we'll discuss later.

Problem 7.1. Prove that every rational number is algebraic.

Numbers that are not algebraic are called *transcendental*. For example, e is a transcendental number, but this is a little tricky to prove.

Problem 7.2. Draw a Venn diagram to show the relationship between $\mathbb{Q}, \overline{\mathbb{Q}}, \mathbb{R}$, and \mathbb{C} . Give an example number for every region in your diagram (no proof necessary).

Problem 7.3. Show that $\overline{\mathbb{Q}}$ is countable.

Algebraic numbers are cool because they are *the algebraic closure of the rationals*. This is why we write $\overline{\mathbb{Q}}$ and it means the following theorem:

Problem 7.4. (Challenge) Prove that z is algebraic if and only if there exists a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ with *algebraic* coefficients $a_0, \dots, a_n \in \overline{\mathbb{Q}}$ such that z is a root of f .

Notice that the above theorem isn't true for \mathbb{R} ! For \mathbb{C} , this is called the fundamental theorem of algebra. Sets like $\overline{\mathbb{Q}}$ and \mathbb{C} that satisfy this theorem are called *algebraically closed*.