# Group Theory II

Kevin Li

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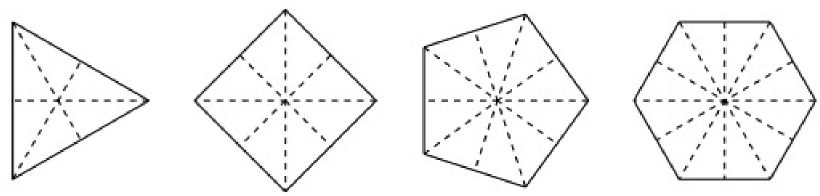
## 1 Warm Up

**Problem 1.** As a reminder, we define $S_n$ as the set of functions $\{f : \{1, \ldots, n\} \to \{1, \ldots, n\} \mid f \text{ is a bijection}\}$. Recall the group axioms from last week. Show that $(S_n, \circ)$ is a group. (Associative, identity, inverse)

Intuitively, since groups represent some sort of symmetry of a certain set of objects, the symmetric group should contain all 'symmetries'. Last time, we saw that $S_n$ is non-abelian. Now, let's look at some other non-abelian groups.

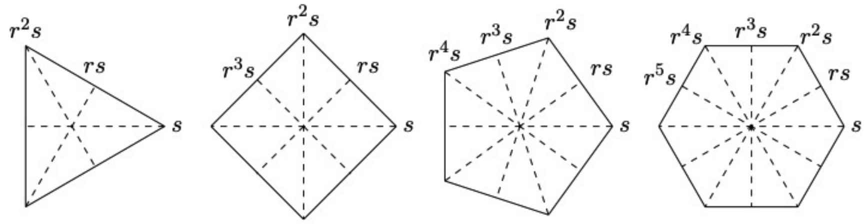
**Example 1.** Suppose you have an equilateral triangle one of the vertices aligned with the x-axis. Let's consider two operations on this triangle:
1. A rotation of 120 degrees counterclockwise
2. A reflection across the x-axis, a 'flip'
Notice how these two operations always keeps one of the vertices aligned with the x-axis. Notice how rotating the triangle 3 times will return to the original triangle. Similarly, if we reflect the triangle two times we will also get the original triangle. We can extend this notion to other regular $n$-gons by changing the angle of rotation to $360/n$ degrees.



**Problem 2.** Using the above example, let $s$ be the reflection operation, and $r$ be the rotation operation. What is the order of $s$ and $r$? (Remember: The order of an element, $g$ is the smallest $k \in \mathbb{N}$ such that $g^k = e$ if such $k$ exists. In other words, how many times should we rotate/flip an $n$-gon to return to the original orientation?)

**Problem 3.** Is this an abelian group? (Hint: Try drawing a picture of an equilateral triangle, labeling vertices, and doing rotations and reflections. Try to find what $sr$ is equal to, where $sr$ defines a rotation and then a reflection)

**Definition 2.** The **Dihedral group** $D_n$ is the set $\{e, r, r^2, \ldots, r^{n-1}, sr, sr^2, \ldots, sr^{n-1}\}$ with multiplication $*$ such that $r^n = e$, $s^2 = e$, and $sr^{n-1} = rs$. Below is a visual representation:




## 2 Group Arithmetic

**Example 3.** Since a group is defined by a set along with a single binary operation, we only can do operations with the one operation. Lets consider some group $(G, *)$. Suppose that $g_1, g_2, g_3 \in G$ and $g_1 * g_2 = g_3$. Then, since inverses exist, then similar to standard operations we can multiply by an inverse element, say $g_1^{-1}$ on the left. **Remember, not all groups are commutative. If you multiply on the left on one side, you have to do the same for the other.** Thus, we can say $g_1^{-1} * g_1 * g_2 = g_1^{-1} * g_3$. Then, it follows that $g_2 = g_1^{-1} * g_3$. Often, we will omit the $*$ symbol, as it is implied as the only operation available.

**Problem 4.** Consider the group $D_n$ as defined above. Show that $srs^{-1} = r$. (Hint: what is $(sr)^2$? Think about it geometrically, then show it mathematically)

**Definition 4.** The **Quaternion Group** or **Dicyclic Group** is the set $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ with multiplication $*$ such that $a^4 = e$, $b^2 = a^2$, and $aba = b$.

**Problem 5.** Fill out the Cayley Table for the Quaternion Group:

| | $e$ | $a$ | $a^2$ | $a^3$ | $b$ | $ab$ | $a^2b$ | $a^3b$ |
|---|---|---|---|---|---|---|---|---|
| $e$ | | | | | | | | |
| $a$ | | | | | | | | |
| $a^2$ | | | | | | | | |
| $a^3$ | | | | | | | | |
| $b$ | | | | | | | | |
| $ab$ | | | | | | | | |
| $a^2b$ | | | | | | | | |
| $a^3b$ | | | | | | | | |

**Definition 5.** Suppose $(G, *)$ is a group. Suppose $g \in G$. Define **Conjugation by g** on some element $x \in G$ by $gxg^{-1}$. Formally, we can define a function $\Phi : G \to G$ where $\Phi_g(x) = gxg^{-1}$.

**Definition 6.** Suppose $(G, *)$ is a group. A **Normal Subgroup of G** is a subgroup $N \leq G$ such that for any $g \in G$ and for any $n \in N$, $gng^{-1} \in N$.

**Problem 6.** Suppose $(G, *)$ is a group, and $N_1, N_2$ are normal subgroups of $G$. Then, show that $N_1 \cap N_2$ is a normal subgroup of $G$. ($N_1 \cap N_2 = \{x \in G \mid x \in N_1 \text{ and } x \in N_2\}$)

1. First, let $x \in N_1 \cap N_2$. How do we show for any $g \in G$, $gxg^{-1} \in N_1$?

2. Proceed similar to show $gxg^{-1} \in N_2$ and conclude $gxg^{-1} \in N_1 \cap N_2$

**Problem 7.** Let $(G, *)$ be an abelian group. Show that any subgroup $H \leq G$ is a normal subgroup.

**Definition 7.** Let $(G, *)$ be a group, and let $H \leq G$. The **Normalizer** or **Centralizer** of $H$ in $G$ is the set $C_H(G) = \{x \in G \mid \text{for any } h \in H, xh = hx\}$. In other words, the set $C_H(G)$ is the set of elements in $G$ that commute with all elements in $H$.

**Problem 8.** Let $(G, *)$ be a group, and let $H \leq G$. Show that $C_H(G)$ is a subgroup of $G$.

**Problem 9. (CHALLENGE)** Let $(G, *)$ be a group, and let $H \leq G$ be a normal subgroup. Show that $C_H(G)$ is a normal subgroup of $G$.

## 3 Quotients

Recall that last week, we defined the set $\mathbb{Z}/n$ as the set $\{[0], [1], \ldots, [n-1]\}$ where $[k]$ is the set of integers congruent to $k \pmod{n}$. Let's formalize this type of association of elements.

**Example 8.** It is a bit difficult to formally define a binary relation without knowledge of set theory. It is included below for anyone interested. Informally, a binary relation is some way to associate elements to one another. For example, define the relation $R$ by $xRy \iff n \mid x-y$. Then, if $k \in \mathbb{Z}$ and $kRx$, it follows that $k-x = n*m$ for some $m \in \mathbb{Z}$. Thus, $k \equiv x \pmod{n}$.

**Definition 9. (BONUS DEFINITION)** A **Binary Relation** R over sets $X, Y$ is a set of ordered pairs $\langle x, y\rangle$ where $x \in X$, $y \in Y$.

**Definition 10.** An **Equivalence Relation** is a binary relation R on a set $X$ that is **reflexive, symmetric**, and **transitive**. In other words,

1. For any $x \in X$, $xRx$ (reflexive)

2. For any $x, y \in X$, $xRy \implies yRx$ (symmetric)

3. For any $x, y, z \in X$, $xRy$ and $yRz$ implies $xRz$ (transitive)

The **Equivalence Class** of an element $x$ is the set $[x]_R = \{y \in X \mid xRy\}$

**Problem 10.** Show that the binary relation $R$ defined on $\mathbb{Z}$ by $xRy \iff n \mid x-y$ is an equivalence relation.

**Definition 11.** Let $(G, *)$ be a group, and $R$ an equivalence relation on $G$. Then, the **Quotient Group** of G by R is the pair $(G/R, *)$ where $G/R = \{[x]_R \mid x \in G\}$ is the set of equivalence classes of G under R.

**Problem 11.** How can we think of quotient groups as describing symmetries of a set? For example, how does $\mathbb{Z}/n$ show some symmetry of $\mathbb{Z}$? (Try to write out the integers in a circle where the numbers in the same equivalence class are at the same point)

## 4 BONUS SECTION

**Definition 12.** Let $(G, *)$ be a group, and $H \leq G$. Define the **Left Cosets** of $H$ in $G$ by a fixed $g \in G$ by $gH = \{gh \mid h \in H\}$. The **Right Cosets** of $H$ in $G$ are defined similarly, $Hg = \{hg \mid h \in H\}$

**Problem 12.** Write explicitly the left and right cosets of $H = \langle b \rangle$ of the Quaternion Group $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$

**Definition 13.** Let $(G, *)$ be a group, $H \leq G$. Define the **set of left cosets of H** denoted by $G/H = \{gH \mid g \in G\}$. The **induced group operation** on $G/H$ is defined by $(xH) * (yH) = (x * y)H$.

**Theorem 14.** *Let $(G, *)$ be a group, $H \leq G$. Consider $G/H = \{gH \mid g \in G\}$ the set of left cosets of $H$. Then, the binary operation $* : G/H \times G/H \to G/H$ is **well defined** (if $xH = x'H$ and $yH = y'H$ then $xyH = x'y'H$) $\iff$ $H$ is normal in $G$.*

**Problem 13. (CHALLENGE)** Prove Theorem 14.