Week 8: Irrational Numbers

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1 Basic Definitions

Today we investigate irrational numbers. Recall that we have the set of integers:

$$..., -3, -2, -1, 0, 1, 2, 3, ...$$

Two positive integers are said to be **coprime** if their greatest common divisor is 1. A real number r is called **rational** if $r = \frac{p}{q}$ for some integers p, q with q nonzero; r is called **irrational** if otherwise. Notice that if a positive rational number $r = \frac{p}{q}$ for positive integers p, q, then we may assume that p, q are coprime by dividing out the greatest common divisor of p and q.

Exercise 0: Is 2022 rational? What about -0.4? What about $\frac{5}{2}$?

Example 1: $\sqrt{2}$ is irrational.

Proof: if $\sqrt{2} = \frac{p}{q}$, where p, q are coprime positive integers, then $p^2 = 2q^2$, hence p is even. Let $p = 2m$, then $2m^2 = q^2$, then q is even. This contradicts the fact that p, q are coprime.

Exercise 2: Which of the following numbers are irrational?
(a) $\sqrt[3]{2}$,
(b) $\sqrt{2} + \sqrt{3}$
(c) $\log_{10} 2$.
(d) $\sqrt{2 + 2\sqrt{2}}$.

Exercise 3: Prove that if a real number can be expressed as $p + q\sqrt{2}$ where p, q are rational, then it has a unique such expression.

We state the following theorem without proof. However, if there is time, you can try to prove it by assuming $x = \frac{m}{n}$ for $m, n \in \mathbb{Z}$, then multiply by n^k.

Theorem 4: In general, if $p(x)$ is a polynomial of the form $p(x) = a_k x^k + a_{k-1} x^{k-1} + ... + a_1 x^1 + a_0$, where a_i, $i = 0, ..., k$ are integers, $a_n \neq 0$, then if $\frac{m}{n}$ is a root of $p(x) = 0$, where m, n are integers such that $|m|$ and $|n|$ are coprime and $n \neq 0$, then m divides a_0 and n divides a_k.

Exercise 5: Prove that the $x^7 - 4x - 1 = 0$ has no real rational roots.

However, in general determining if something is rational or not is extremely difficult; famously it is still open whether $\pi + e, \pi - e, \pi \cdot e, \ln \pi$ are irrational or not.

2 Approximating Irrationals by Fractions

Irrationals can be approximated by rationals. For example, $\sqrt{2} = 1.414213...$, so we can have the following approximation

$$1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}...$$

However, these are bad approximations. For example, you can check that $\left|\frac{17}{12} - \sqrt{2}\right| < 0.00246$, however $\left|\frac{141}{100} - \sqrt{2}\right| = 0.0042...$, so $\frac{17}{12}$ is a much better approximation to $\sqrt{2}$ than $\frac{141}{100}$, even though the first one has much smaller denominator (12) than that of the second one (100). Another problem is that a priori we don't know the decimal expansions of $\sqrt{2}$, hence we cannot get approximations like $1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}...$ without first use a calculator to see what $\sqrt{2}$ is. A third problem is that we would like to have some sort of error estimation on our approximation.

The way to proceed is as follows:

$0 < \sqrt{2} - 1 < \frac{1}{2}$, this is obvious.

Square both sides: $0 < 3 - 2\sqrt{2} < \frac{1}{4}$,

this rearranges to $0 < \frac{3}{2} - \sqrt{2} < \frac{1}{8}$! Something miraculous has happened, start with an approximation with error bound $\frac{1}{2}$, we got an approximation with error bound $\frac{1}{8}$!

Square $0 < 3 - 2\sqrt{2} < \frac{1}{4}$ one more time, we get $0 < 17 - 12\sqrt{2} < \frac{1}{16}$, this rearranges to $0 < \frac{17}{12} - \sqrt{2} < \frac{1}{192}$! What a ridiculously efficient way of computing $\sqrt{2}$.

Exercise 6: Find a fraction that approximates $\sqrt{2}$ to within 10^{-20}. Do the same with $\sqrt{5}$. Surprisingly, you don't need many iterations to achieve such high accuracy.

Just how good is our result precisely? The following argument shows that this is the best possible.

Example 7: Recall $\frac{17}{12}$ approximates $\sqrt{2}$ to within 0.00246, if $\frac{p}{q}$ with p, q coprime is another fraction that approximates $\sqrt{2}$ to within 0.00246, then from the triangle inequality, we have:

$$\left|\frac{p}{q} - \frac{17}{12}\right| \leq \left|\frac{p}{q} - \sqrt{2}\right| + \left|\frac{17}{12} - \sqrt{2}\right| \leq 0.00246 + 0.00246 = 0.00492.$$

However, $\left|\frac{p}{q} - \frac{17}{12}\right| = \left|\frac{12p - 17q}{12q}\right| \geq \frac{1}{12q}$, hence $\frac{1}{12q} \leq 0.00492$, so $q \geq 16.9...$, so $q \geq 17$! i.e., if you want a fractional that approximates $\sqrt{2}$ better than $\frac{17}{12}$, you need one with denominator at least 17!

Exercise 8: Repeat the process above for π and $\frac{355}{113}$. You may use the fact that $\frac{355}{113}$ approximates π to within 2.67×10^{-7}. You may be in for a surprise.

The process above is related to continued fractions. Our approximations to $\sqrt{2}$ obtained by keep squaring stuff can also be obtained by using continued fractions. It is possible to show that continued fractions give the best approximations possible. Similarly, $\frac{355}{113}$ can be obtained by using continued fraction as well. The topic of approximating reals by rationals is called **Diophantine approximation**.

Exercise 9: Modify our method to obtain rational approximation to $\sqrt[3]{2}$.[1]

[1] The same iterative procedure can also be obtained from Newton's method.