MARTINGALES

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Warning: In order to reduce the amount of overhead required many technical points are swept under the rug and some places are not fully rigorous!

1. Probability

We start with a review of probability. The most basic example is flipping a fair coin.
We have some set Ω which is called the sample space. In the case of flipping a coin, $\Omega = \{H, T\}$ which stand for heads and tails. Next we have the space of events denoted $\mathcal{F}$. These are subsets of the sample space that we can assign probabilities to.
In the case of the coin we have $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$.

Problem 1.1. What is the sample space and the space of events for flipping two coins?

Finally we have a probability measure P which takes events in $\mathcal{F}$ and assigns them a probability, for example $P(\{H\}) = \frac{1}{2}$ and $P(\emptyset) = 0$.

Problem 1.2. What is the event for flipping one head and one tail? What is its probability?

You might be wondering why we make such a fuss about writing $\mathcal{F}$ when it seems like $\mathcal{F}$ is just the powerset of Ω. However we will see shortly (the next section) why we may not want $\mathcal{F}$ to be the entirety of the powerset of Ω.
A random variable is a function $X : \Omega \to \mathbb{R}$. We write $E(X)$ for the expected value of X which is $\sum_{i \in \Omega} X(i)P(\{i\})$.

Problem 1.3. Set $X = 1$ when we get heads and $X = 0$ when we get tails. What is $E(X)$?

For an event A we define $E(X|A)$ (the expectation of X given A) to be

$$\frac{\sum_{i \in A} P(\{i\})X(i)}{P(A)}$$

This can be thought of as the best guess for X given that the event A occurs.

Problem 1.4. What is $E(X|\Omega)$?

Problem 1.5. If X is the number of heads after two coin flips and A is the event that the first coin flip is heads what is $E(X|A)$?

2. Stochastic Processes

We start with some vague intuition before moving to vague definitions. This worksheet deals with stochastic processes which are random variables that change over time. For this worksheet time will always be discrete. Specifically a stochastic processes is a sequence of random variables indexed by natural numbers.
A martingale is a stochastic processes that is "fair". The example to keep in mind is the following. Suppose at every second we flip a coin. Let X_n be the number of heads minus the number of tails after n seconds. We have $X_0 = 0$ and at time n, X_n is just as likely to go up as it is to go down (this is what is meant by fair).
An important notion in stochastic processes is that of information, particularly what information is available at what time. For example, we restrict to the case where we have 3 coins. Now we will have $\Omega = \{H, T\}^3$. Now imagine that we are at time 1 second. We can see the first coin flip but we have no idea what will happen in the future. We now ask what events are there so that by just seeing the first coin flip we can be sure whether or not that event will come to pass.

Problem 2.1. What are these events?

Problem 2.2. Interpret the events obtained in the last problem.

We call this collection of sets $\mathcal{F}_1$ which should be though of as the information available at time 1. We define $\mathcal{F}_n$ similarly. In this sense we use collections of sets to encode "information".

Problem 2.3. In the setting of the coin flips describe what $\mathcal{F}_n$ is? How big is it?

Problem 2.4. Prove that if A, B are disjoint events so that $E(X_{n+1}|A) = E(X_n|A)$ and $E(X_{n+1}|B) = E(X_n|B)$ then $E(X_{n+1}|A \cup B) = E(X_n|A \cup B)$.

Definition 2.1. $\mathcal{G}$ is called a generator of $\mathcal{F}$ if every event in $\mathcal{F}$ is a disjoint union of events in $\mathcal{G}$.

Problem 2.5. Find the smallest generator you can for $\mathcal{F}_n$.

Finally we have

Definition 2.2. X_n is a martingale if $E(X_{n+1}|A) = E(X_n|A)$ for any $A \in \mathcal{F}_n$.

In light of the above problem in order to verify that X_n is a martingale we only need verify the definition for events in a generator.

Problem 2.6. Prove that X_n is a martingale if X_n is the difference between the number of heads and tails after n flips.

Problem 2.7. Let X_n be the number of heads up to time n. Is this a martingale?

Problem 2.8. Prove that X_n is a martingale if $-X_n$ is.

Problem 2.9. Let X_n be a martingale. Prove that $E(X_n) = E(X_0)$.

Problem 2.10. Find a stochastic process X_n so that $E(X_n) = E(X_0)$ for all n but X_n isn't a martingale.

3. Slick Solutions

One neat trick we can do with martingales is to bet on them.
Imagine it as follows. At every second Alice tosses a coin. Again X_n is the number of heads minus number of tails. However before each coin toss Bob is allowed to place a bet, B_n. If the next coins is heads he gets B_n dollars otherwise he loses B_n dollars. Another way to say this is that Bob gets $B_n(X_{n+1} - X_n)$ dollars. This is what is meant by betting on the martingale X_n. The total amount of money Bob has won or lost up to (but not including) time n is denoted S_n. In particular $S_0 = 0$.
The main point is that B_n needs to be a predictable process in that B_n can only depend on information available up to time $n - 1$. Bob can't see the coin before betting on it.

Proposition 3.1. *S_n is a martingale if X_n is a martingale.*

Finally we consider the possibility that Bob might not play a fixed number of rounds. For example Bob might decide to stop playing the moment he wins his first bet.
The final ingredient we need is the optional stopping theorem:

Theorem 3.2. *Suppose that Bob comes up with any betting strategy. If the expected amount of time that Bob plays is finite then the expected amount of money that Bob wins is 0.*

Now our strategy to solve problems is to cook up a martingale, bet on it, and use that $E[S_n] = 0$.

Problem 3.1. What is the expected amount of time to flip two heads in a row?

Problem 3.2. What is the expected amount of time to flip k heads in a row?

Problem 3.3. What is the expected amount of time to flip a heads followed by a tails. Try to do this without martingales first.

Problem 3.4. What is the expected amount of time to see HTH?

Problem 3.5. Describe a (reasonable) procedure to determine the expected amount of time to see any particular sequence of flips.

4. Bonus: To infinity and beyond

Consider the situation where we are allowed to flip the coin as many times as we want. Suppose that Bob bets one dollar on each coin flip being heads and gets up the moment he is in the black (i.e. $S_n > 0$).

Problem 4.1. What is the probability that Bob gets up at some finite time?

Problem 4.2. How much money has Bob made/lost when he gets up?

Problem 4.3. Does this contradict the optional stopping theorem?

Betting can also help us prove neat theorems.

Problem 4.4. Suppose that X_n is a martingale that never grows bigger than some fixed constant K. Show that X_n converges as $n \to \infty$.

Problem 4.5. (Challenge) Prove Proposition 3.1.

Problem 4.6. Prove the optional stopping theorem under the assumption that Bob's strategy never takes more than M seconds, where M is some fixed constant.