Week 8: Irrational Numbers

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1 Basic Definitions

Today we investigate irrational numbers. Recall that we have the set of integers:

..., −3, −2, −1, 0, 1, 2, 3, ...

Two positive integers are said to be coprime if their greatest common divisor is 1. A real number \( r \) is called rational if \( r = \frac{p}{q} \) for some integers \( p, q \) with \( q \) nonzero; \( r \) is called irrational if otherwise. Notice that if a positive rational number \( r = \frac{p}{q} \) for positive integers \( p, q \), then we may assume that \( p, q \) are coprime by dividing out the greatest common divisor of \( p \) and \( q \).

Exercise 0: Is 2022 rational? What about \(-0.4\)? What about \(\frac{5}{2}\)?

Example 1: \( \sqrt{2} \) is irrational.

Proof: if \( \sqrt{2} = \frac{p}{q} \), where \( p, q \) are coprime positive integers, then \( p^2 = 2q^2 \), hence \( p \) is even. Let \( p = 2m \), then \( 2m^2 = q^2 \), then \( q \) is even. This contradicts the fact that \( p, q \) are coprime.

Exercise 2: Which of the following numbers are irrational?

(a) \( \sqrt{2} \)
(b) \( \sqrt{2} + \sqrt{3} \)
(c) \( \log_{10} 2 \)
(d) \( \sqrt{2} + 2\sqrt{2} \).

Exercise 3: Prove that if a real number can be expressed as \( p + q\sqrt{2} \) where \( p, q \) are rational, then it has a unique such expression.

We state the following theorem without proof. However, if there is time, you can try to prove it by assuming \( x = \frac{m}{n} \) for \( m, n \in \mathbb{Z} \), then multiply by \( n^k \).

Theorem 4: In general, if \( p(x) \) is a polynomial of the form \( p(x) = a_kx^k + a_{k-1}x^{k-1} + \ldots + a_1x + a_0 \), where \( a_i, i = 0, ..., k \) are integers, \( a_n \neq 0 \), then if \( \frac{m}{n} \) is a root of \( p(x) = 0 \), where \( m, n \) are integers such that \( |m| \) and \( |n| \) are coprime and \( n \neq 0 \), then \( m \) divides \( a_0 \) and \( n \) divides \( a_k \).

Exercise 5: Prove that the \( x^7 - 4x - 1 = 0 \) has no real rational roots.

However, in general determining if something is rational or not is extremely difficult; famously it is still open whether \( \pi + e, \pi - e, \pi \cdot e, \ln \pi \) are irrational or not.
2 Approximating Irrationals by Fractions

Irrationals can be approximated by rationals. For example, $\sqrt{2} = 1.414213...$, so we can have the following approximation:

$$\frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}...$$

However, these are bad approximations. For example, you can check that $|\frac{17}{12} - \sqrt{2}| < 0.00246$, however $|\frac{141}{100} - \sqrt{2}| = 0.0042...$, so $\frac{17}{12}$ is a much better approximation to $\sqrt{2}$ than $\frac{141}{100}$, even though the first one has much smaller denominator (12) than that of the second one (100). Another problem is that a priori we don’t know the decimal expansions of $\sqrt{2}$, hence we cannot get approximations like $\frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}...$ without first use a calculator to see what $\sqrt{2}$ is. A third problem is that we would like to have some sort of error estimation on our approximation.

The way to proceed is as follows:

$$0 < \sqrt{2} - 1 < \frac{1}{2}$$

Square both sides: $0 < 3 - 2\sqrt{2} < \frac{1}{4}$,

this rearranges to $0 < \frac{3}{2} - \sqrt{2} < \frac{1}{4}$! Something miraculous has happened, start with an approximation with error bound $\frac{1}{2}$, we got an approximation with error bound $\frac{1}{8}$!

Square $0 < 3 - 2\sqrt{2} < \frac{1}{4}$ one more time, we get $0 < 17 - 12\sqrt{2} < \frac{1}{16}$, this rearranges to $0 < \frac{17}{12} - \sqrt{2} < \frac{1}{192}$! What a ridiculously efficient way of computing $\sqrt{2}$.

**Exercise 6**: Find a fraction that approximates $\sqrt{2}$ to within $10^{-20}$. Do the same with $\sqrt{5}$. Surprisingly, you don’t need many iterations to achieve such high accuracy.

Just how good is our result precisely? The following argument shows that this is the best possible.

**Example 7**: Recall $\frac{17}{12}$ approximates $\sqrt{2}$ to within 0.00246, if $\frac{p}{q}$ with $p, q$ coprime is another fraction that approximates $\sqrt{2}$ to within 0.00246, then from the triangle inequality, we have:

$$\left|\frac{p}{q} - \frac{17}{12}\right| \leq \left|\frac{p}{q} - \sqrt{2}\right| + \left|\frac{17}{12} - \sqrt{2}\right| \leq 0.00246 + 0.00246 = 0.00492.$$

However, $\left|\frac{p}{q} - \frac{17}{12}\right| = \frac{|12p - 17q|}{12q} \geq \frac{1}{12q}$, hence $\frac{1}{12q} \leq 0.00492$, so $q \geq 16.9...$, so $q \geq 17$! i.e., if you want a fractional that approximates $\sqrt{2}$ better than $\frac{17}{12}$, you need one with denominator at least 17!

**Exercise 8**: Repeat the process above for $\pi$ and $\frac{355}{113}$. You may use the fact that $\frac{355}{113}$ approximates $\pi$ to within $2.67 \times 10^{-7}$. You may be in for a surprise.

The process above is related to continued fractions. Our approximations to $\sqrt{2}$ obtained by keep squaring stuff can also be obtained by using continued fractions. It is possible to show that continued fractions give the best approximations possible. Similarly, $\frac{355}{113}$ can be obtained by using continued fraction as well. The topic of approximating
reals by rationals is called **Diophantine approximation**.

**Exercise 9:** Modify our method to obtain rational approximation to $\sqrt[3]{2}$.\(^1\)

\(^1\)The same iterative procedure can also be obtained from Newton's method.