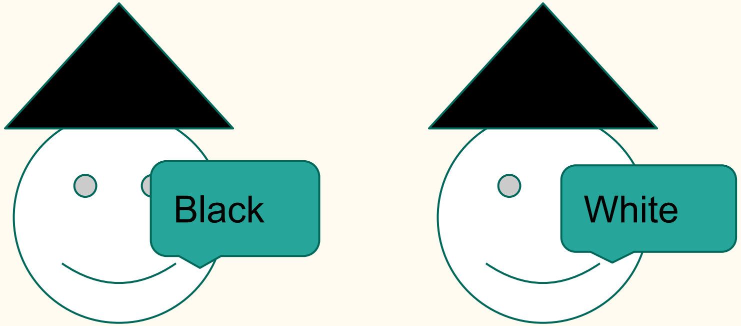
Hat puzzles

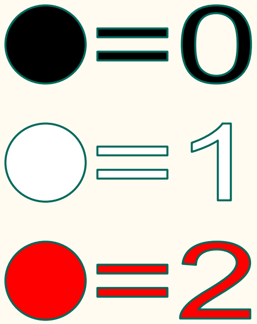
The Mathematics of Coordinated Inference

2 prisoners 2 colors

There are 2 prisoners. Each prisoner is assigned a random hat, either black or white. The prisoners can see the hats of each other but not their own. Now, they must each, simultaneously, say only one word which must be "black" or "white". If the word matches their hat color they are released, and if one prisoner resume their liberty they can rescue the other. They can communicate beforehand. How can they get free?

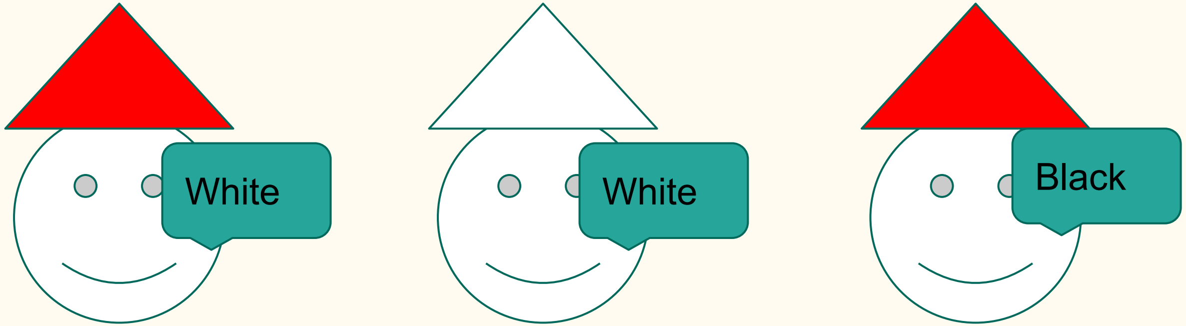


Three prisoners, three colors



Now for three prisoners the hats can be black, white or red.

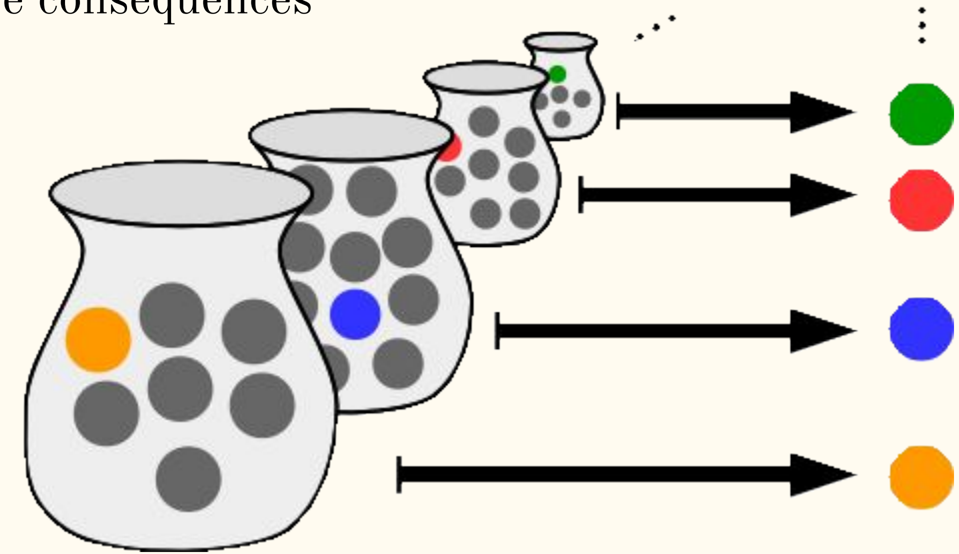
Can one of the prisoners guess their hat color?



Yes! Prisoner i assumes that sum of colors mod 3 is equal to i.

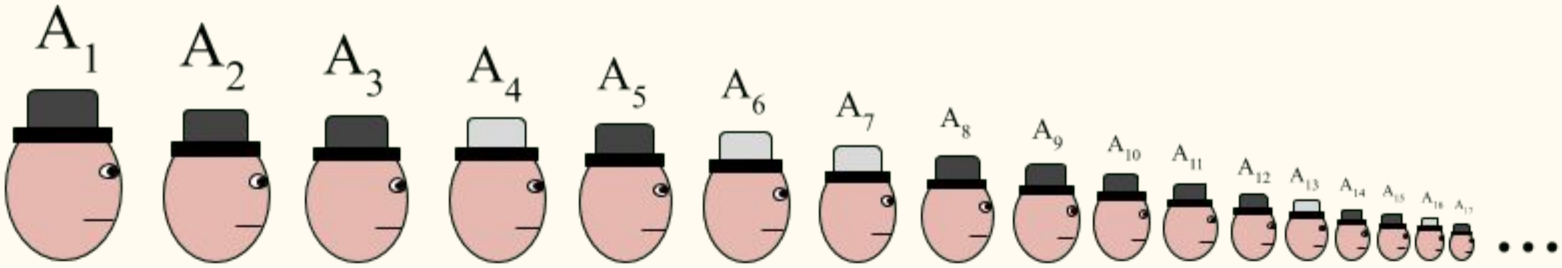
Axiom of choice

- Intuitive
- Has unintuitive consequences

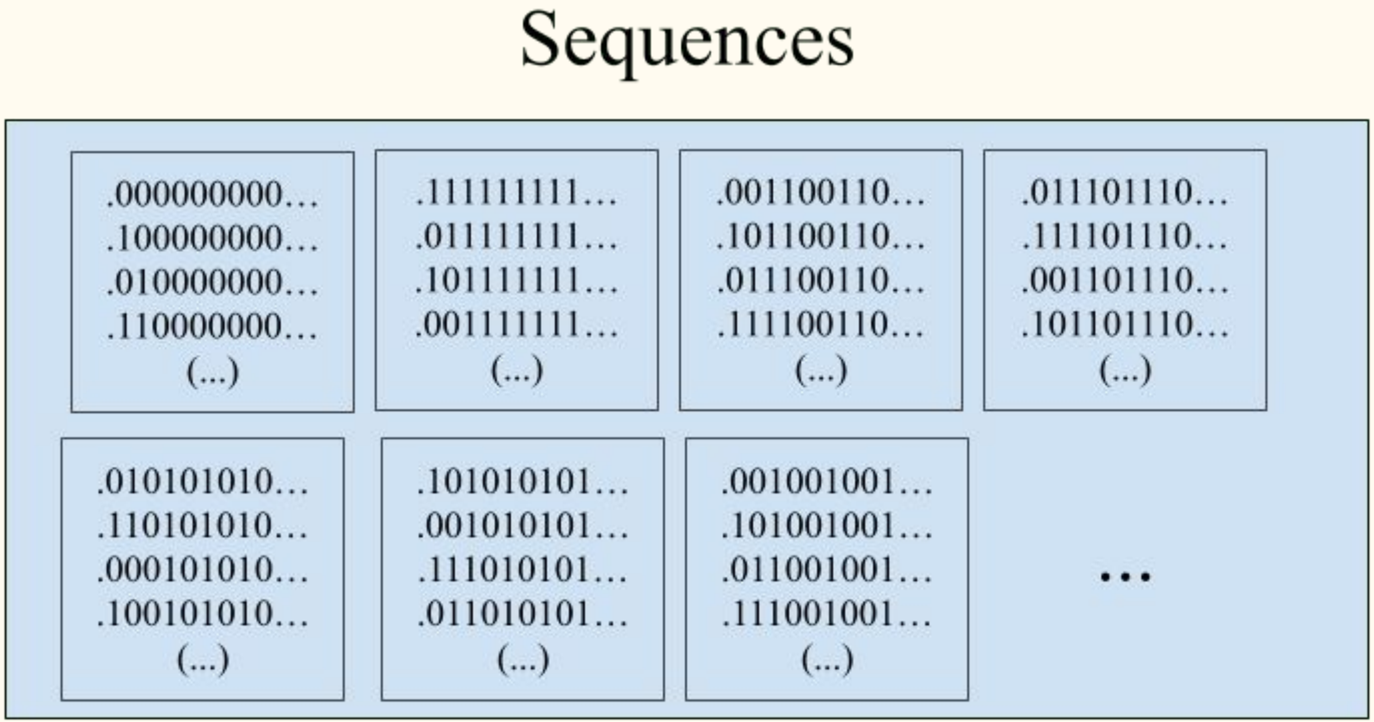


Infinitely many prisoners in a row

There are a countable infinity of prisoners lined up with randomly assigned hats. They each know their position in line. The reward for a right guess is freedom and the punishment for a wrong guess is death. Can you find a strategy that ensures that only finitely many prisoners are killed?

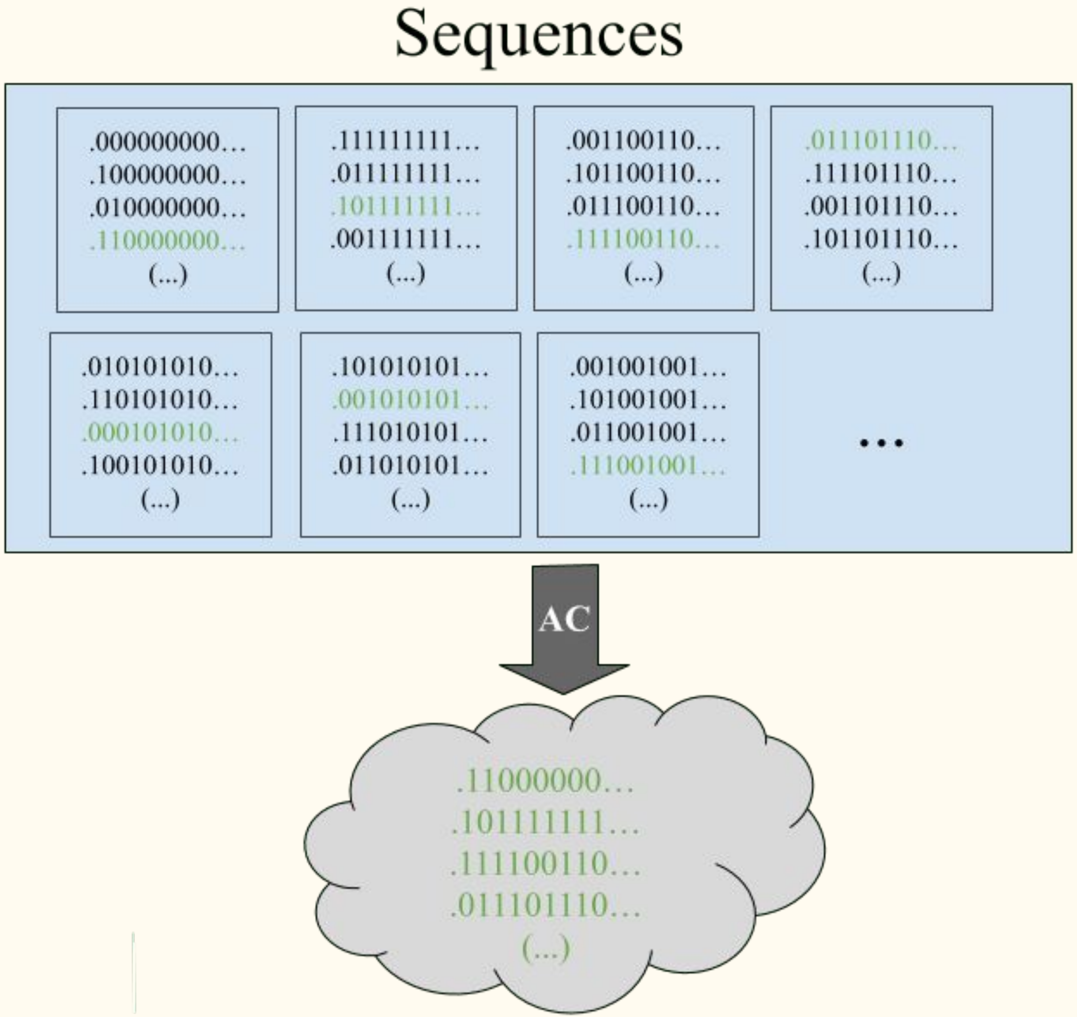


Solution:


Sequences
.000000000…
.100000000…
.010000000…
.110000000…
(...)
.111111111…
.011111111…
.101111111…
.001111111…
(...)
.001100110…
.101100110…
.011100110…
.111100110…
(...)
.011101110…
.111101110…
.001101110…
.101101110…
(...)
.010101010…
.110101010…
.000101010…
.100101010…
(...)
.101010101…
.001010101…
.111010101…
.011010101…
(...)
.001001001…
.101001001…
.011001001…
.111001001…
(...)
…

Solution:

- Prisoners agree on the choice function in advance
- Each prisoner understands the equivalence class they are in
- Each prisoner names the hat color it would have in the green sequence



Different infinities

We say that two sets have the same size if they have a one-to-one correspondence.

Which of these sets have the same size?

$\mathbb{N}$ – set of natural numbers

$2^{\mathbb{N}}$ – set of all subsets of natural numbers

$2\mathbb{Z}$ – set of even integers

$\mathbb{R}$ – set of real numbers

$\mathbb{Z}$ – set of integers

$\mathbb{N}^2$ – set of all pairs of natural numbers

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$2^{\mathbb{N}}$ – set of all subsets of natural numbers

$\mathbb{R}$ – set of real numbers

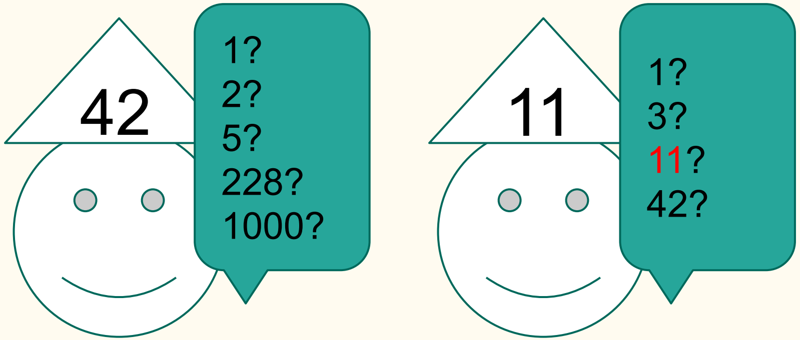
Continuum hypothesis

CH: There is no set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$.

- First Hilbert's problem is to prove CH.
- "Resolved" in 1963 by Paul Cohen.
- The answer to this problem is independent of ZFC, so that either the continuum hypothesis or its negation can be added as an axiom to ZFC set theory, with the resulting theory being consistent if and only if ZFC is consistent.

2 prisoners $|\mathbb{N}|$ colors

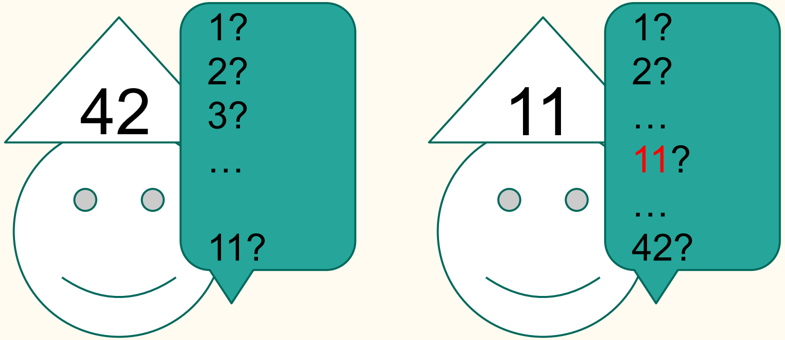
Now assume that there are 2 prisoners and their hats contain natural numbers. Each prisoner is allowed to come up with finite list of guesses. How can they ensure that at least 1 guess is correct?



Solution: 2 prisoners $|\mathbb{N}|$ colors

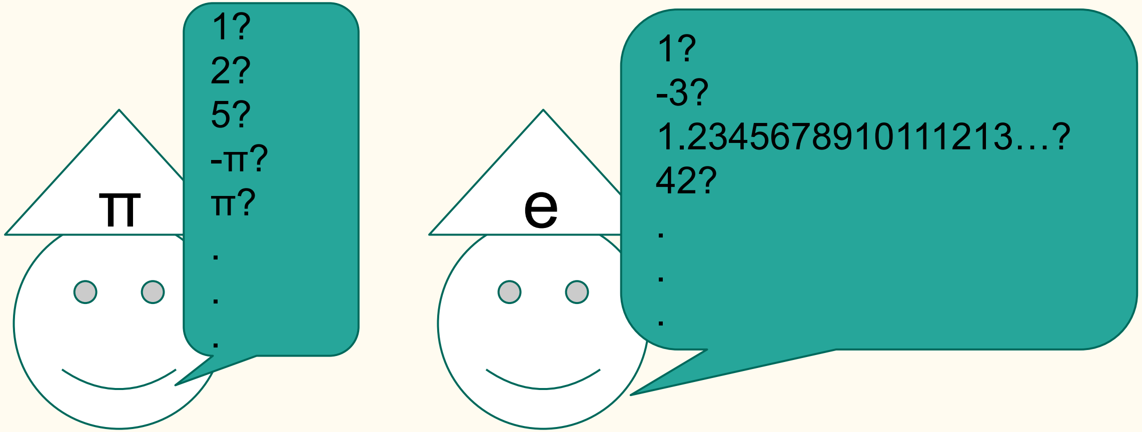
Now assume that there are 2 prisoners and their hats contain natural numbers. Each prisoner is allowed to come up with finite list of guesses. How can they ensure that at least 1 guess is correct?

Guess the numbers less than or equal to the number on the other prisoners hat.



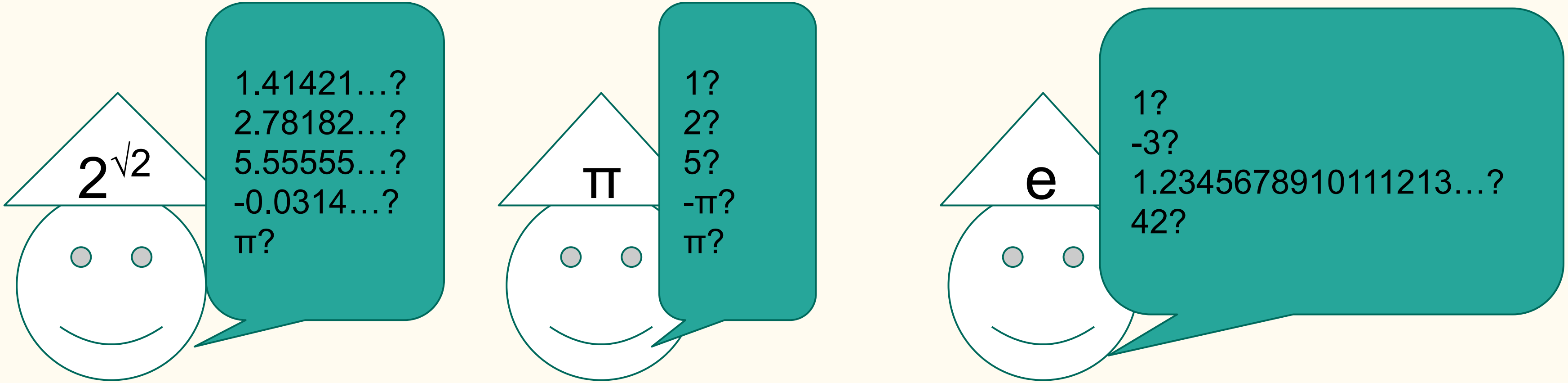
2 prisoners $|\mathbb{R}|$ colors

Now assume that there are 2 prisoners and their hats contain **real** numbers. Each prisoner is allowed to come up with **countable** list of guesses. How can they ensure that at least 1 guess is correct?



3 prisoners $|\mathbb{R}|$ colors, assuming CH

Now assume that there are **3** prisoners and their hats contain **real** numbers. Each prisoner is allowed to come up with **finite** list of guesses. How can they ensure that at least 1 guess is correct?



Solution:

Use the **well-ordering principle!**

- Well-order definition: A well-order of a set is a total order where every non-empty subset has a least element.

  Not well-ordered:
  -1, -0.5, -0.25, ..., 0, ⇦... 0.125, 0.25, 0.5, 1

  Well-ordered:
  -1, -0.5, -0.25, ..., 0, 1, 0.5, 0.25, ...

- Well-ordering principle: Every set can be well ordered.
- The well ordering principle is a theorem of ZFC and a formulation of the axiom of choice.

4.8, 5.8, 39. 85..., e, ..., π, ... , 47. 581...., ..., 12.5, ... , ...,15, ...

Solution: Two prisoners, countably many guesses

Prisoners agree on a well ordering of the real numbers, $<_a$.

Case 1: For every real number r,

$\{x: x <_a r\}$ is countable.

Each prisoner will guess all the numbers up to the number on the other prisoner's hat.

2.736,

8.278…,

…..,

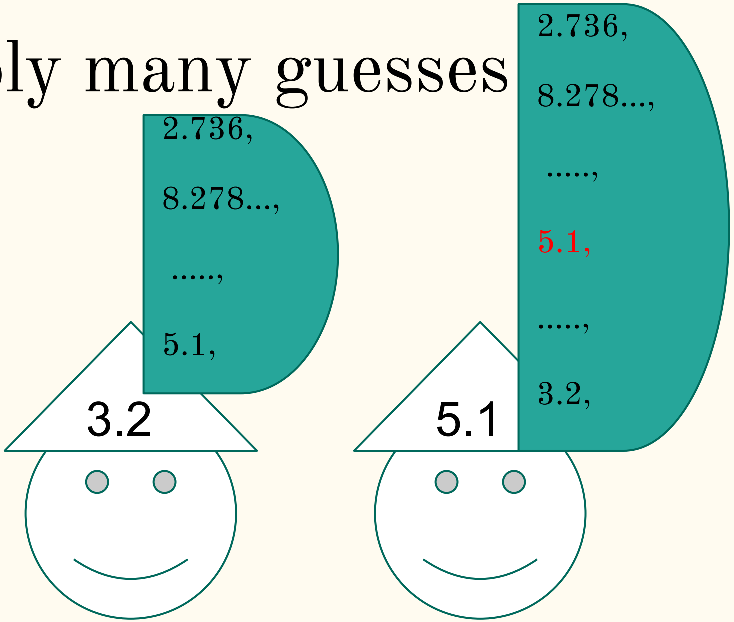
5.1,

…..,

3.2,

……,

……



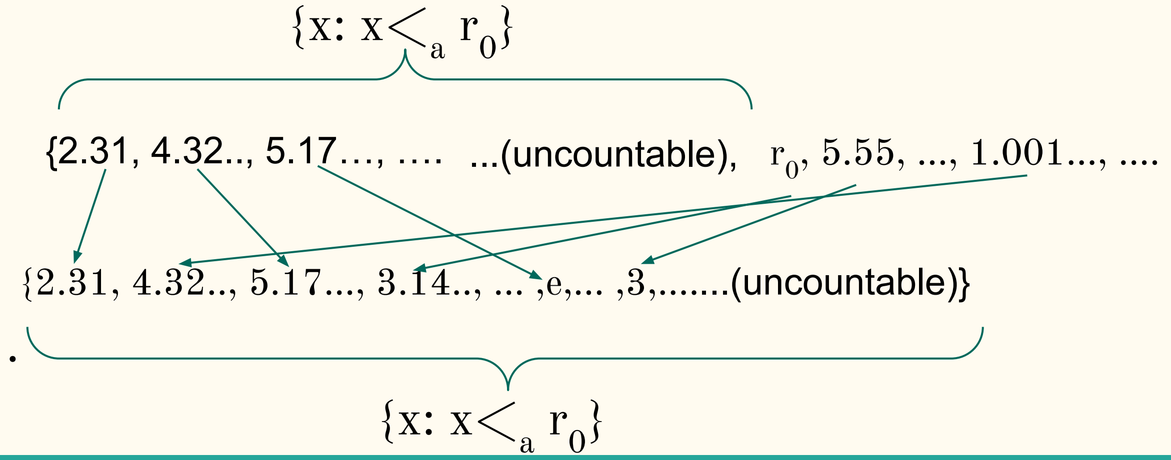
-Numbers are listed right after all the numbers before them in the well ordering

Solution: Two prisoners, countably many guesses

Prisoners agree on a well ordering of the real numbers, $<_a$.

Case 2: There is a real number r where $\{x: x <_a r\}$ is uncountable.

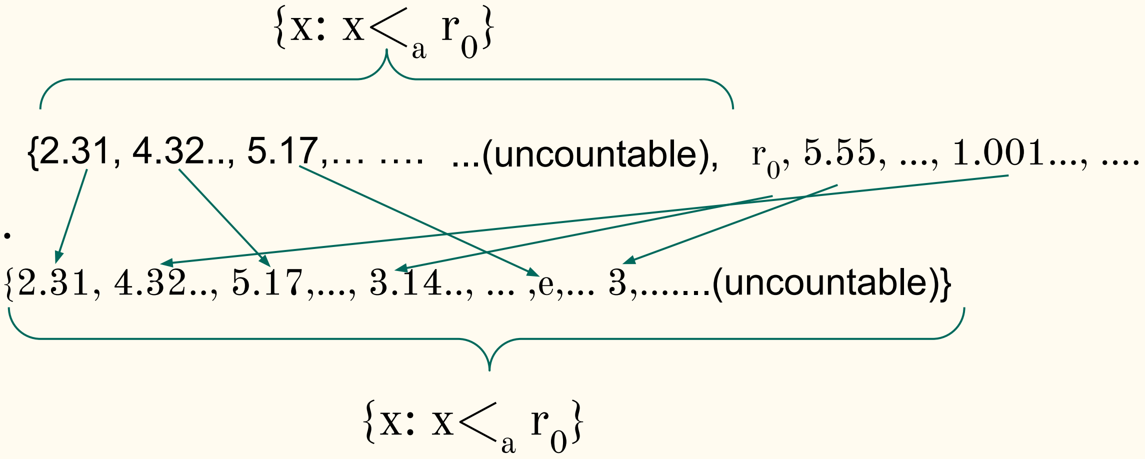
There will be a least (according to the order) r_0 that satisfies that property. By the continuum hypothesis, the prisoners can agree on a bijection between $\{x: x <_a r_0\}$ and $\mathbb{R}$.



Solution: Two prisoners, countably many guesses

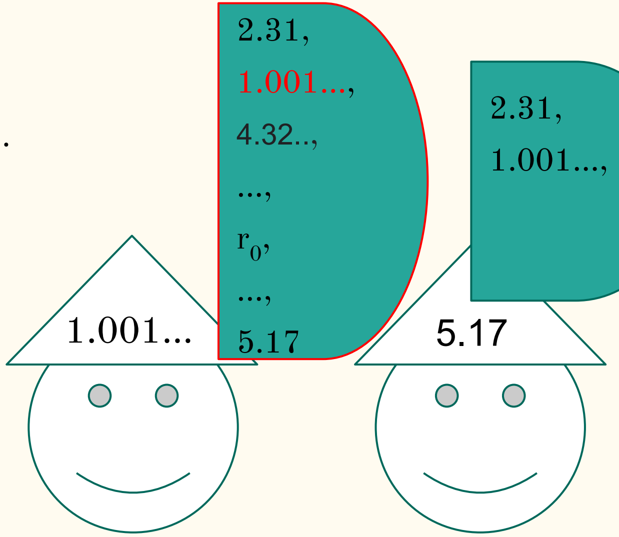
Prisoners agree on a well ordering of the real numbers, $<_a$.

Case 2: We use the bijection to get to case 1.



2.31
1.001...
4.32..,
...
r_0,
...
5.17
...
5.55

......

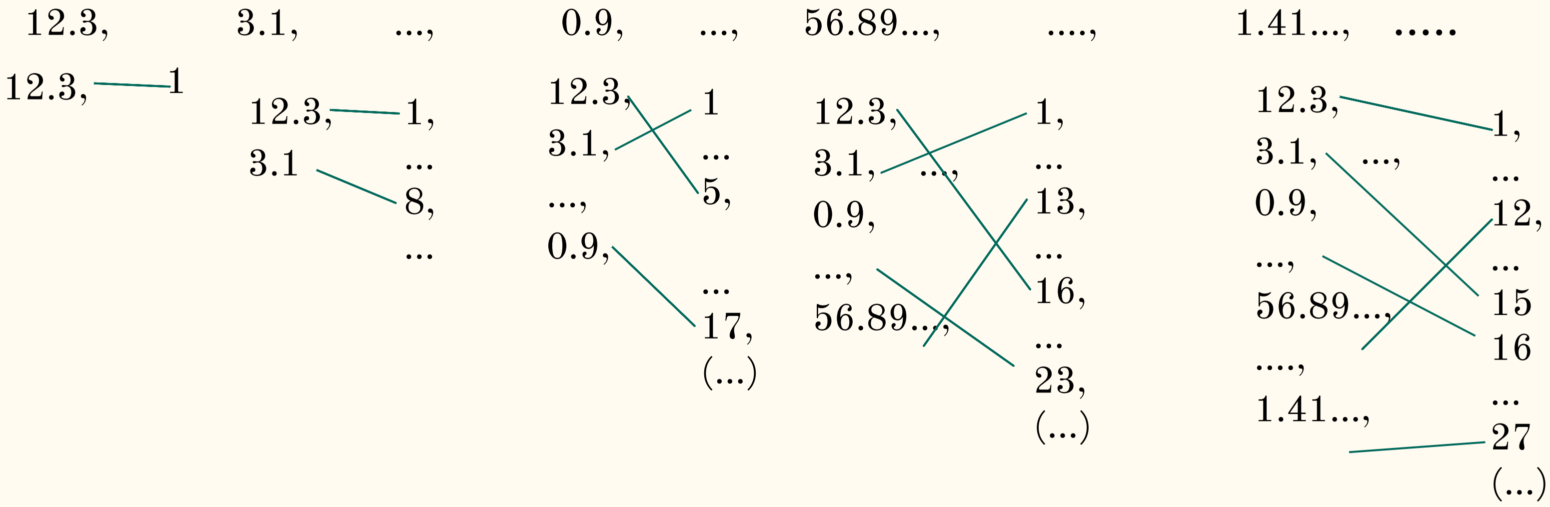


-Numbers are listed right after all the numbers befor
them in the well ordering

Solution: Three prisoners, finitely many guesses

We will use the axiom of choice even more:

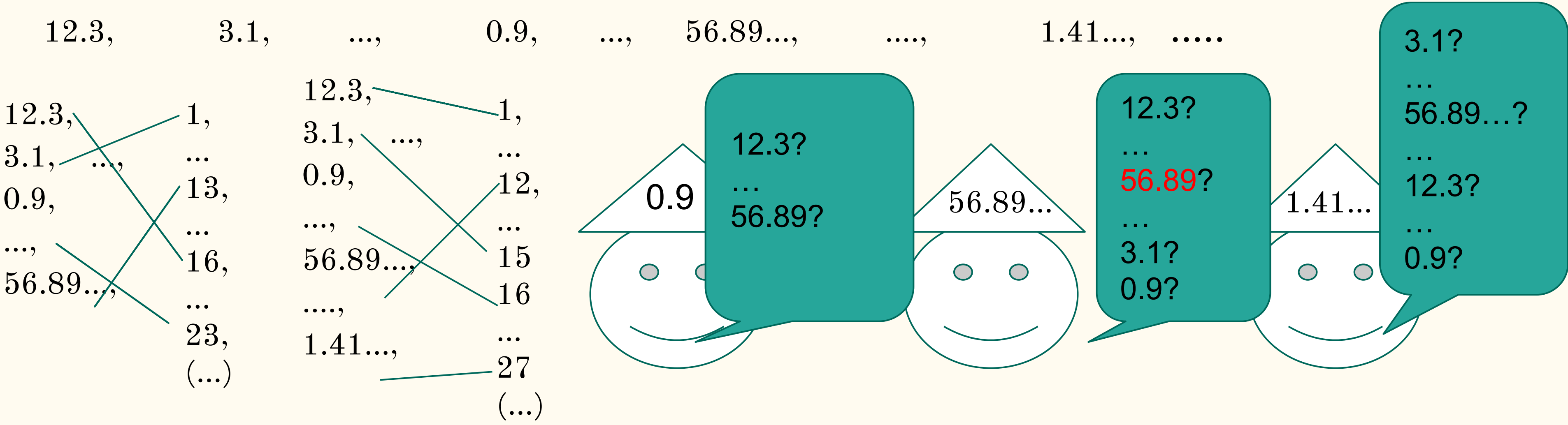
The prisoners will agree on a well ordering of $\mathbb{R}$, $<_{a,}$ where each real number is greater than only countably many real numbers. For each real number r, the prisoners will also agree on a injection between the numbers less than or equal to r (according to the well order) and $\mathbb{N}$.



Solution: Three prisoners, finitely many guesses

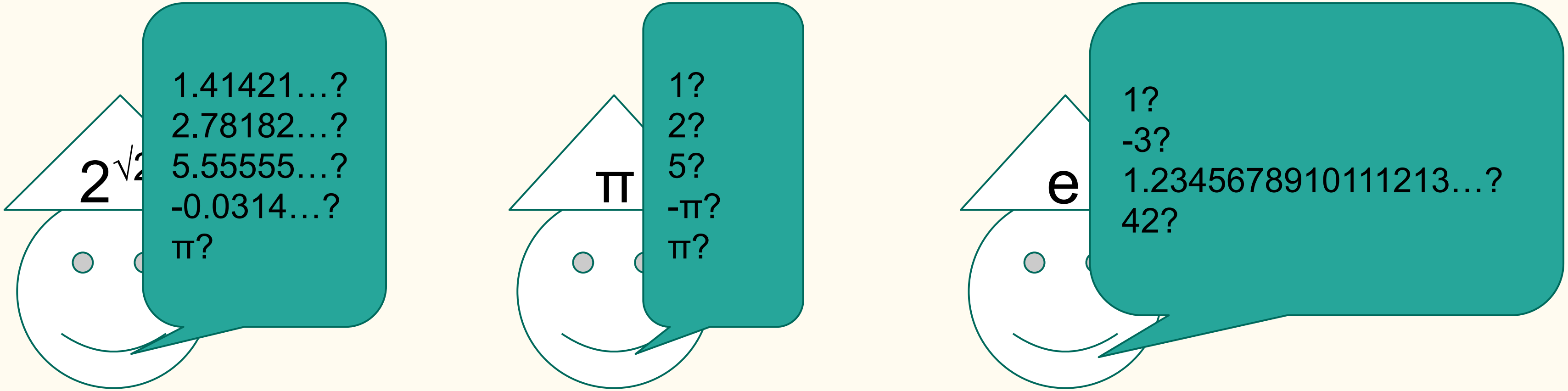
We will use the axiom of choice even more:

Strategy: The prisoners will look at the maximum of the two other hats in $<_a$. Next they will look at the chosen bijection for that hat and guess until the other number.



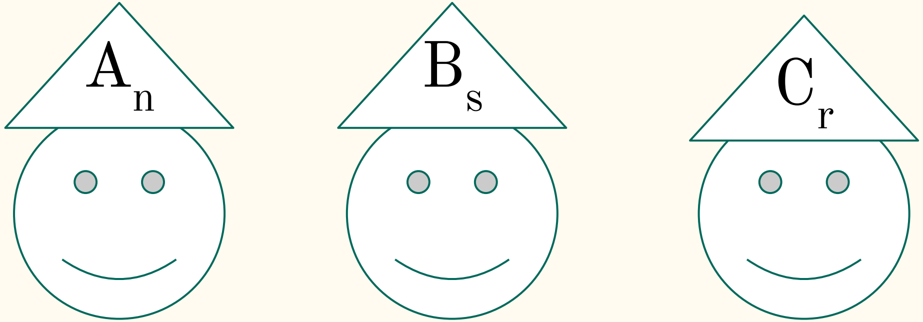
3 prisoners $|\mathbb{R}|$ colors, NOT assuming CH

Now assume that there are **3** prisoners and their hats contain **real** numbers. Each prisoner is allowed to come up with **finite** list of guesses. How can they ensure that at least 1 guess is correct?



Solution

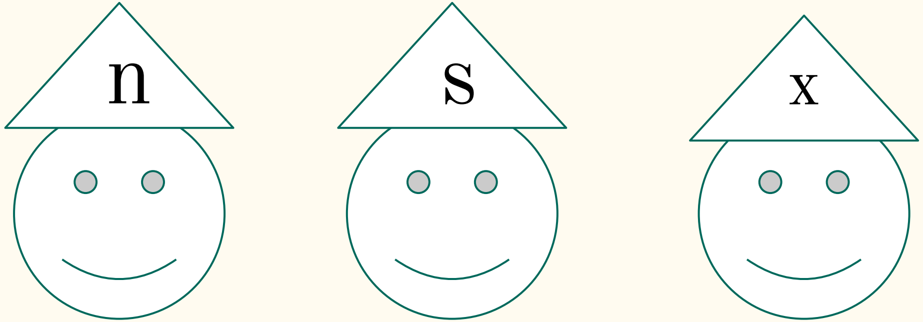
Suppose $|\mathbb{N}| < |S| < |\mathbb{R}|$ for some $S \subseteq \mathbb{R}$. If the prisoners have a winning strategy, it will work for the game when we restrict the hat numbers they have.



Solution

Suppose $|\mathbb{N}|<|S|<|\mathbb{R}|$ for some $S\subseteq\mathbb{R}$ and that the prisoners have a strategy.

We will find an (n, s, x), such that using the given strategy, no prisoner guesses their hat color.



$|\{(a, b, c)\text{: prisoner 3 guesses correctly in } (a, b, c)\text{:}\}|\leq|\mathbb{N}| \times |S| \times |\mathbb{N}|=|S|<|\mathbb{R}|$ so we can find an x that prisoner 3 never guesses.

$|\{(a, b)\text{: prisoner 2 guesses correctly in } (a, b, x)\}|\leq\{|\mathbb{N}| \times |\mathbb{N}|=|\mathbb{N}|<S$ which similarly lets us pick s.

A has finitely many guess so there is some n where if s is on the second prisoners hat and, x is on the third prisoners hat n is not guessed.