Polynomials I - The Cubic Formula

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Adapted from worksheets by Oleg Gleizer.

1 Cubic Equations by Long Division

Definition 1 *A* ***cubic polynomial*** *(cubic for short) is a polynomial of the form* $ax^3 + bx^2 + cx + d$*, where* $a \neq 0$.

The *Fundamental Theorem of Algebra* (which we will not prove this week) tells us that all cubics have three roots in the complex numbers. Recall:

Definition 2

- *The* ***rectangular form*** *of a complex number is* $a + bi$*, where* a *is the* ***real part*** *and* b *(not* bi*!) is the* ***imaginary part****.*
- *The* ***polar form*** *of a complex number is* $re^{i\theta}$*, where* r *is the* ***modulus*** *and* θ *is the* ***argument****. Note that two arguments which differ by an integer multiple of* 2π *give the same complex number.*
- *These two forms of complex numbers are related by:*

$$a = r\cos(\theta) \text{ and } b = r\sin(\theta)$$
$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

The first method of solving cubics will be for the simplest case. As an example, let's see how we would divide $x^3 + 3x^2 + 5x - 4$ by $x - 1$. Similarly to the usual long division, we multiply the *divisor* by the most simple thing (a monomial) such that when we subtract the result, the *leading terms* of the polynomial cancel. We then repeat until we get a *quotient* and a *remainder*, like so:

$$\begin{array}{r|l}
 & x^2 + 4x + 9 \\ \hline
x - 1 & x^3 + 3x^2 + 5x - 4 \\
 & -x^3 + x^2 \\ \hline
 & 4x^2 + 5x \\
 & -4x^2 + 4x \\ \hline
 & 9x - 4 \\
 & -9x + 9 \\ \hline
 & 5
\end{array}$$

In this case, we obtain a *quotient* of $x^2 + 4x + 9$ and a *remainder* of 5 - in general, the remainder is always lower degree than the divisor. In order to use this method to solve cubics, we will need to first **find one root** of our cubic. Once we have found one root (say, r), there will be no remainder after long dividing by $x - r$. To see this, let's work through an example.

Problem 1 *Let's find all the roots of* $x^3 - 5x - 2$.

- Find one root of $x^3 - 5x - 2$. (Hint: Guess and check - it's an integer.)

  Solution: -2 is a root.

- Perform long division to divide $x^3 - 5x - 2$ by $x - r$, where r is the root you found.

  Solution:

$$\begin{array}{r l} & x^2 - 2x - 1 \\ x+2 \big) & x^3 \qquad\;\; - 5x - 2 \\ & -x^3 - 2x^2 \\ \hline & \quad -2x^2 - 5x \\ & \qquad 2x^2 + 4x \\ \hline & \qquad\quad -x - 2 \\ & \qquad\qquad x + 2 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

- Find the other two roots. (Hint: You should have gotten a quadratic from the previous part. How does one solve a quadratic?)

  Solution: By the quadratic formula, the other roots are $1 + \sqrt{2}$ and $1 - \sqrt{2}$.

The main flaw of this method is that it requires us to be able to look at a cubic and guess one of its roots. Unfortunately, that's not always so easy. Thankfully, we have another method to find one root, from Tartaglia (but named after Cardano - long story!)

2 Depressed Cubics and Tartaglia's Method

Definition 3 *A cubic is said to be in* ***depressed form*** *if its leading coefficient is* 1 *and its second coefficient is* 0. *In other words, a cubic in depressed form is written* $x^3 + px + q$.

Problem 2 *Show that any cubic can be brought to depressed form.*

- Start with a general cubic $ax^3 + bx^2 + cx + d$. Why can we assume that $a = 1$?

  Solution: Because a is not zero, whatever it is we can divide the entire polynomial by it, keeping the roots the same.

- In order to depress the cubic $x^3 + bx^2 + cx + d$, we make the *substitution* $x = y - \alpha$. Find the correct value for α to make the resulting cubic depressed.

  Solution: Plug in $x = y - \alpha$ and expand everything:

$$(y - \alpha)^3 + b(y - \alpha)^2 + c(y - \alpha) + d = y^3 + (-3\alpha + b)y^2 + \text{other terms}$$

  so that $\alpha = b/3$ is the correct value.

Problem 3 *Given a depressed cubic $x^3 + px + q$:*

- Make the substitution $x = u + v$, and expand out everything.

  Solution:

$$x^3 + px + q = u^3 + 3u^2v + 3uv^2 + v^3 + pu + pv + q$$

- In order to find the roots, let us set the above equation equal to zero. Try factoring out a $u + v$ (but in a way such that you don't just get your original equation back!) and write down some relationships between u, v, p, q.

  Solution: We can group the $3u^2v + 3uv^2$ on the left-hand side:

$$0 = u^3 + 3u^2v + 3uv^2 + v^3 + pu + pv + q = (u^3 + v^3 + q) + (3uv + p)(u + v)$$

  Since $u + v = x$, we can treat the right-hand side as a linear polynomial in x:

$$0 = (3uv + p)x + (u^3 + v^3 + q) \text{ so that}$$
$$3uv + p = 0 \text{ and } u^3 + v^3 + q = 0$$

- Using the relations we found above, rewrite the equation in terms of only u. Making the substitution $y = u^3$, solve the resulting equation for u.

  Solution: Solving our first relation for v, we find that $v = -p/3u$. Plugging this in, we obtain

$$0 = u^3 + q - \frac{p^3}{27u^3} + \text{terms that cancel}$$

  Multiplying both sides by u^3 to get rid of the denominator and substituting $y = u^3$ gives

$$y^2 + qy - \frac{p^3}{27} = 0$$

  so by the quadratic formula,

$$u = \sqrt[3]{y} = \sqrt[3]{\frac{-q \pm \sqrt{q^2 - 4(-p^3/27)}}{2}}$$

  Since we only need one root, we take one such value of u; without loss of generality the one with the plus:

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

- Solve for v and write down a root of $x^3 + px + q$. (The correct answer is on the next page for your reference.)

  Solution: Using our other relation $u^3 + v^3 + q = 0$, we obtain

$$v = \sqrt[3]{-q - u^3} = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

We should have seen that

Theorem 1 *(Cardano's Formula) Given a depressed cubic x^3+px+q, one of its roots is given by*

$$\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

Problem 4 *Use Cardano's Formula to find one root of the following polynomials.*

- x^3+6x-2

  Solution:

  $$x=\sqrt[3]{4}-\sqrt[3]{2}$$

- x^3+6x^2+9x-2 (Hint: You will need to bring this to the depressed form first.)

  Solution: Bring it to the depressed form by substituting $x=y-2$:

  $$x^3+6x^2+9x-2=(y-2)^3+6(y-2)^2+9(y-2)-2=y^3-3y-4$$

  and then using Cardano's Formula gives

  $$x=\sqrt[3]{2+\sqrt{3}}+\sqrt[3]{2-\sqrt{3}}$$

Problem 5 *Use Cardano's Formula to find a root of the polynomial x^3-5x-2 from earlier. Which root did you find?*

Solution: Cardano's Formula gives

$$x=\sqrt[3]{1+\sqrt{\frac{-98}{27}}}+\sqrt[3]{1-\sqrt{\frac{-98}{27}}}$$

Here u and v are not real numbers, and clearly u^3 and v^3 are complex conjugates, so u and v must also be complex conjugates. Therefore

$$x=u+v=2\mathrm{Re}(u)$$

Clearly u has positive real and imaginary parts, so therefore $\mathrm{Re}(u)\geq\mathrm{Re}(u^3)$ (students should think geometrically why this is true!), so the real part of u is at least 1 and therefore $x\geq 2$. But the only root of x^3-5x-2 which is larger than 2 is $1+\sqrt{2}$, so in this case Cardano's Formula gave the root $1+\sqrt{2}$.

3 Roots of Unity and the General Cubic Formula

As you may have noticed, Cardano's Formula is a very inconvenient way to solve a cubic, as you will now need to divide the cubic by $x - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. That's usually not an easy task, but thankfully our study of complex numbers (recall from last quarter!) will help us simplify the process.

Problem 6

- *What are **all** arguments of* 1*?*

  Solution: $2\pi k$ *for all integers* k*.*

- *Find **all*** n^{th} *roots (in polar form) of* 1*, ie **every** complex* z *such that* $z^n = 1$*. These are called the* n^{th} ***roots of unity**.*

  Solution: If $z^n = 1$ *then* $z^n = e^{2\pi ik}$ *for some integer* k*, so that* $z = e^{2\pi ik/n}$ *for some integer* k*. Since* k *and* $k + n$ *give the same* z*, we therefore must have only* n *different roots, corresponding to* $k = 0, ..., n - 1$*.*

- *Graph the* 3^{rd} *roots of unity in the complex plane. Then the* 4^{th}*,* 5^{th}*, etc. until you see a pattern. What pattern do you notice?*

  Solution: The n^{th} *roots of unity form a regular* n*-gon with a vertex at* 1*.*

Definition 4 *An n^{th} root of unity z is called* ***primitive*** *if $z^m \neq 1$ for all $m = 1, ..., n-1$. Primitive roots of unity are usually denoted ζ.*

Problem 7 *Prove that an n^{th} root of unith $e^{2\pi ik/n}$ is primitive if and only if k and n are relatively prime.*

Solution: If k and n are not relatively prime, let $d > 1$ be a common divisor. Then $0 < n/d < n$ and $(e^{2\pi ik/n})^{n/d} = e^{2\pi ik/d} = 1$, so that $e^{2\pi ik/n}$ is not primitive. Conversely, if k and n are relatively prime, then km is not divisible by n for any $0 < m < n$, so that $e^{2\pi ik/n}$ is primitive.

In the case of cubics, 1 and 2 are both relatively prime to 3, so we can pick either corresponding root of unity to be ζ.

Problem 8 *Write ζ (either one) in rectangular form.*

Solution: $\zeta = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Problem 9 *In terms of ζ, find all three solutions to $x^3 = c$, for any real number c.*

Solution: $x = \sqrt[3]{c}, x = \zeta\sqrt[3]{c}, x = \zeta^2\sqrt[3]{c}$

Problem 10 • *Using Problem 9, return to Problem 3. In the step where you solved for u, solve for **all three** possible values of u.*

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, u = \zeta\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, u = \zeta^2\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

- *Prove that for a depressed cubic $x^3 + px + q$, **all three** of its roots are given by*

$$x_1 = u + v, x_2 = \zeta u + \zeta^2 v, x_3 = \zeta^2 u + \zeta v$$
$$\textit{where } u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \textit{ and } v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

*This is the **general cubic formula**.*

Solution: Using the equation $u^3 + v^3 = q$, we see that

$$v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

but since $3uv = -p$ and p is real while ζ is not, this means that uv has to have no ζ or ζ^3. Therefore, for every choice of u there is the unique choice of v which gives a real number, and this gives the three roots in the form above.

Problem 11 *Use the formula from Problem 10 to find all three roots of the following polynomials.*

- $x^3 + 6x - 2$

  Solution: Simplifying all the expressions gives

$$x = \sqrt[3]{4} - \sqrt[3]{2}, x = \left(\frac{\sqrt[3]{2} - \sqrt[3]{4}}{2}\right) + \left(\frac{\sqrt{3}(\sqrt[3]{2} + \sqrt[3]{4})}{2}\right) i, x = \left(\frac{\sqrt[3]{2} - \sqrt[3]{4}}{2}\right) + \left(\frac{-\sqrt{3}(\sqrt[3]{2} + \sqrt[3]{4})}{2}\right) i$$

- $x^3 + 6x^2 + 9x - 2$ (Hint: You will need to bring this to the depressed form first.)

  Solution: Recall the depressed form from Problem 4 and simplifying gives

$$x = \sqrt[3]{2 + \sqrt{3}} + \sqrt[3]{2 - \sqrt{3}}, x = \left(\frac{-\sqrt[3]{2 + \sqrt{3}} - \sqrt[3]{2 - \sqrt{3}}}{2}\right) + \left(\frac{\sqrt{3}(\sqrt[3]{2 + \sqrt{3}} - \sqrt[3]{2 - \sqrt{3}})}{2}\right) i$$
$$, x = \left(\frac{-\sqrt[3]{2 + \sqrt{3}} - \sqrt[3]{2 - \sqrt{3}}}{2}\right) + \left(\frac{\sqrt{3}(\sqrt[3]{2 - \sqrt{3}} - \sqrt[3]{2 + \sqrt{3}})}{2}\right) i$$

Problem 12 *Let's return to our first example,* $x^3 - 5x - 2$*. Use the formula you derived in Problem 10. Which root is* x_1*?* x_2*?* x_3*?*

Solution: We've established in Problem 5 that $x_1 = 1 + \sqrt{2}$. We argue similarly for the other roots. Since u and v are complex conjugates, and ζ and ζ^2 are also complex conjugates,

$$x_2 = \zeta u + \zeta^2 v = 2\mathrm{Re}(\zeta u) \text{ and } x_3 = \zeta^2 u + \zeta v = 2\mathrm{Re}(\zeta^2 u)$$

Since u lies in the first quadrant, $\mathrm{Re}(\zeta^2 u)$ is the most negative (this is clear by drawing the picture), so that since $1 - \sqrt{2} > -2$, $x_3 = -2$ and $x_2 = 1 - \sqrt{2}$.

4 Bonus Section: Discriminants of Cubics

In the case of quadratics, the *discriminant* $b^2 - 4ac$ determines whether the two roots are both real or not. There is a generalized version for any kind of polynomial.

Definition 5 *Let $r_1, ..., r_n$ be the roots of a degree n polynomial $p(x) = a_n x^n + ... + a_0$. Then the discriminant of p is given by*

$$\Delta = a_n^{2n-2} \prod_{i<j} (r_i - r_j)^2$$

Problem 13 *In terms of the three roots r_1, r_2, r_3, give the formula for the discriminant of a cubic.*

Solution: $\Delta = a_3^4 (r_1 - r_2)^2 (r_1 - r_3)^2 (r_2 - r_3)^2$

Let's classify cubics based on their discriminants.

Problem 14 *Suppose that a complex number z is a root of a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$. Show that its conjugate $\overline{z}$ is also a root of p.*

Solution: Let z be a root. Then since a, b, c, d are real, they are their own complex conjugates and so

$$\overline{0} = \overline{az^3 + bz^2 + cz + d} = a\overline{z}^3 + b\overline{z}^2 + c\overline{z} + d$$

so that $\overline{z}$ is also a root of p.

Problem 15 *Given a cubic p and its discriminant Δ, prove that:*

- *If $\Delta = 0$, then p has a repeated root.*

  Solution: If $\Delta = 0$ then one of $(r_1 - r_2)$, $(r_1 - r_3)$, and $(r_2 - r_3)$ is zero since a_3 is not zero, so those two roots are the same.

- *If $\Delta > 0$, then p has three real roots.*

  Solution: Since a_3 is real (and not zero), $a_3^4 > 0$, so $(r_1 - r_2)^2(r_1 - r_3)^2(r_2 - r_3)^2 > 0$. The roots have to be distinct, otherwise the right-hand side would be zero. By Problem 14, nonreal roots come in conjugate pairs, so if not all the roots are real, then there are exactly 2 non-real roots, and without loss of generality call them r_1 and r_2. Now since r_3 is real,

  $$(r_1 - r_3)^2(r_2 - r_3)^2 = [(r_1 - r_3)(\overline{r_1 - r_3})]^2 = |r_1 - r_3|^2 > 0$$
  $$\textit{and } (r_1 - r_2)^2 = [2iIm(r_1)]^2 = -4Im(r_1) \leq 0$$

  so that $\Delta < 0$, which is a contradiction. Therefore p has three distinct real roots.

- *If $\Delta < 0$, then p has one real root and two non-real roots.*

  Solution: If r_1, r_2, r_3 are real, then clearly $\Delta > 0$, so that if $\Delta < 0$, then p has a non-real root. As discussed previously, this means it has one real root and two non-real roots which are conjugates.

Problem 16 *For each cubic below, is its discriminant positive, negative, or zero?*

- $x^3 - 5x - 2$

  Solution: Positive

- $x^3 + 6x - 2$

  Solution: Negative

- $x^3 - 3x^2 + 3x - 1$

  Solution: Zero