

Olga Radko Math Circle Competition

Advanced 3

Week 5 Winter 2022

1. BONUS PROBLEMS

Each problem is worth 2 points unless otherwise noted. Questions that are marked as 4 points are generally harder.

Unless otherwise noted problems may be attempted as many times as you want without any penalty, however this may be changed at the discretion of the instructor.

In general problems with a numerical solution will be given full points for just the correct numerical value however this is up to the discretion of the instructor (especially for problems with easily guessed numerical answers).

Problem 1.1. Say that Alice flips a coin (possibly indefinitely) until she obtains both one heads and one tails. What is the expected value of the number of flips she will do?

Problem 1.2. Prove that if $x + \frac{1}{x} = 2 \cos(\theta)$, then $x^n + \frac{1}{x^n} = 2 \cos(n\theta)$.

Problem 1.3. Which is larger, 100^{300} or $300!$? The answer is worth 1 point, you get one try. The proof is worth the other point.

Problem 1.4. Prove using the content of previous worksheets that if $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$, then the determinant of

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is the volume of the parallelepiped with corners at $0, u, v, w, u + v, u + w, v + w, u + v + w$.

Problem 1.5. Use the cross product and the magnitude function (but not the dot product) to find a formula for the shortest distance from the point v to the line through the origin and the point w in 3D space.

Problem 1.6. In a team of guards, each is assigned a different positive integer. For any two guards, the ratio of the two numbers assigned to them is at least 3:1. A guard assigned the number n is on duty for n days in a row, off duty for n days in a row, back on duty for n days in a row, and so on. The guards need not start their duties on the same day. Is it possible that on any day, at least one in such a team of guards is on duty?