Math Circle Advanced 1

Winter quarter 2022 midterm game

1 Algebra (Jiahan)

Problem 1.1.
Let $P(n)$ be a polynomial of degree 3 such that $P(0) = 1$, $P(1) = 4$, $P(2) = 11$ and $P(3) = 28$. What is $P(4)$?

Problem 1.2.
Find all prime numbers p such that $p^2 + 11$ has precisely 6 divisors, including 1 and itself.

Problem 1.3.
We call a positive integer n nice, if the sum of all digits of 5^n is 2^n. Find all nice n.

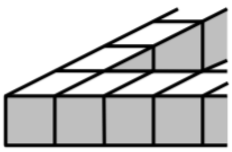
Problem 1.4.
Consider an equation $x^3 - 3x - 1 = 0$ and its three roots x_1, x_2, x_3. Compute $(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2$. You may use the fact that $x^3 - 3x - 1 = 0$ indeed has three real roots.

Problem 1.5.
Compute

$$1 \times 2 \times 3 \times 4 + 2 \times 3 \times 4 \times 5 + ... + 2019 \times 2020 \times 2021 \times 2022.$$

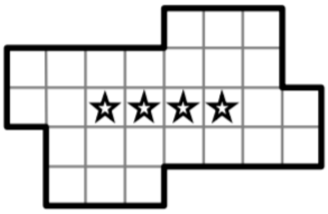
2 Geometry (Nikita)

Problem 2.1.
Masha used 36 equal cubes to build a fence around the square field. Part of this fence is shown in the picture below. How many more cubes will she need to cover the field?

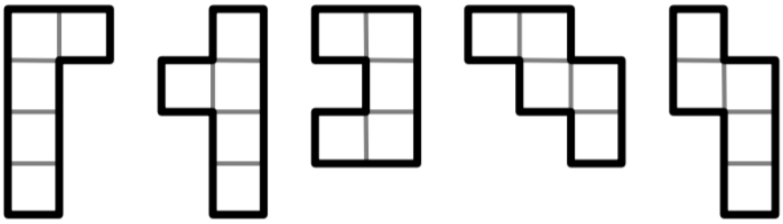


Problem 2.2.
A 300×300 square is divided by red lines into "vertical" 3×2 rectangles, and by blue lines into "horizontal" 23 rectangles. How many individual 1×1 squares will be obtained if cuts are made along all lines?

Problem 2.3.
Cut this figure into 4 parts, so that all parts are equal and each has a star in it.



Problem 2.4.
Make a square out of all the figures below.



Problem 2.5.
Let O be the center of the circumcircle of triangle ABC, with $\angle AOC = 60°$. Find the angle AMC, where M is the center of the circle inscribed in triangle ABC. Be careful, there are 2 possible answers!

3 Combinatorics (Hunter)

Problem 3.1.
In a club there are 5 upperclassmen and 5 alumni. The club is trying to form a committee of 5 members such that:

- All the members are alumni or upperclassmen,
- The committee has a member who is a president,
- The president must also be an alumni.

How many different ways can the committee be chosen?

Problem 3.2.
An inspector is checking electrical circuitry in 4 different buildings, one building per day over the next 8 days. Call the buildings A, B, C, D. The inspector must check the circuitry twice for each building but cannot check the circuitry for buildings A nor B on two consecutive days. How many ways can the inspector visit the buildings?

Problem 3.3.
How many unique 8 letter words are there using exactly 4 X's and 4 Y's such that any leading string of letters always contains more X's than Y's? For example, XXXYYYXY is fine but not XYYXXXYY, because the latter contains XYY as a leading term.

Problem 3.4.
Compute

$$1 \cdot \binom{7}{1} + 3 \cdot \binom{7}{3} + 5 \cdot \binom{7}{5} + \cdots + 7 \cdot \binom{7}{7},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. (Note: ask an instructor for what a binomial coefficient is if needed.)

Problem 3.5.
Compute the following sum:

$$\binom{8}{0}^2 + \binom{8}{1}^2 + \binom{8}{2}^2 + \ldots + \binom{8}{8}^2.$$

4 Probability (Swee Hong)

Problem 4.1.
You can choose exactly one of these two games to play with an instructor:

- A number from 1 to 10 is selected at random. You gain full score of this problem if the number is even, and you gain no point otherwise.
- You play two round of the following game: A number from 1 to 10 is selected at random, and you win the round if the number is divisible by 3. If you win the first round, then you gain full score of this problem. If you lose the first round but win the second round, you gain one half o the score of this problem. If you lose both rounds then you gain no points.

Problem 4.2.
What is the probability that the square of an integer selected at random will end with the digit 1?

Problem 4.3.
The hunter has two dogs. Once, getting lost in the forest, he came to a fork. The hunter knows that each of the dogs will choose the way home with probability 0.7. He decided to let the dogs out one by one. If both choose the same path, he will follow them; if they separate, the hunter will choose the path by tossing a coin. What is the chance that he will choose the right path?

Problem 4.4.
Two hunters Alice and Bob set out to hunt ducks. Each of them hits as often as he misses when shooting at ducks. Alice shoots at 50 ducks during the hunt while Bob shoots at 51. What is the probability that Bob hitting more ducks than Alice?

Problem 4.5.
One hundred people are waiting in line at a movie theater for yet another Avengers' movies. Fifty of them have 10-dollar bills and the other 50 people have only 20-dollar bills. The tickets cost 10 dollars each. When the movie theater opens there is no money in the cashier. If each customer buys just one ticket, what is the probability that none of them will have to wait for change?

5 Theory of Numbers

Problem 5.1.
Let $x_1, x_2, \ldots, x_{2022}$ be 2022 arbitrary real numbers. What is the smallest value the following number can be?

$$\frac{x_1^2 + x_2^2 + \ldots + x_{2022}^2}{(x_1 + x_2 \ldots + x_{2022})^2}$$

Problem 5.2.
Find a four digit number which is a square of an integer, and such that its first two digits are the same and also its last two digits are the same.

Problem 5.3.
The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (that is, 1, 16, 31, etc.). This process is continued until a number is reached which has already been marked. How many unmarked numbers remain?

Problem 5.4.
If the integers from 1 to 222,222,222 are written down in succession, how many 0's are written?

Problem 5.5.
What is the next number in the following sequence?

$$4,\ 14,\ 1114,\ 3114,\ 132114,\ 1113122114,\ 311311222114, \ldots$$