Olga Radko Math Circle Competition

Advanced 3

Week 5 Winter 2022

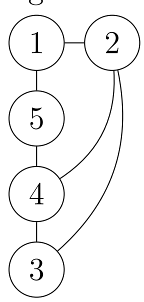
Each problem is worth 2 points unless otherwise noted. Questions that are marked as 4 points are generally harder.
Unless otherwise noted problems may be attempted as many times as you want without any penalty, however this may be changed at the discretion of the instructor.
In general problems with a numerical solution will be given full points for just the correct numerical value however this is up to the discretion of the instructor (especially for problems with easily guessed numerical answers).

1. Spanning trees

Problem 1.1. Let T be a tree with n vertices.

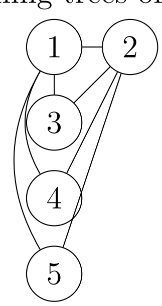
(1) What is the maximum number of leaves T can have?
(2) How many distinct labeled trees with n vertices are there with that many leaves?

Problem 1.2. Find the number of spanning trees of the following graph:



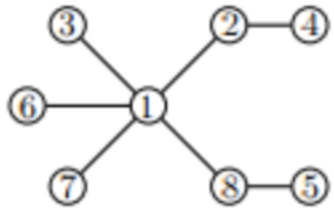
X

Problem 1.3. Find the number of spanning trees of the following graph:



Y

Problem 1.4. Find the Prufer code for the following tree.



Problem 1.5. Build the tree that has Prufer code (3,1,3,3,1,6).

2. Probability

Problem 2.1 (4 points)**.** You can choose exactly one of these two games to play with an instructor:

- A number from 1 to 10 is selected at random. You gain full score of this problem if the number is even, and you gain no point otherwise.
- You play two round of the following game: A number from 1 to 10 is selected at random, and you win the round if the number is divisible by 3. If you win the first round, then you gain full score of this problem. If you lose the first round but win the second round, you gain one half of the score of this problem. If you lose both rounds then you gain no points.

Problem 2.2. Two hunters Alice and Bob set out to hunt ducks. Each of them hits as often as he misses when shooting at ducks. Alice shoots at 50 ducks during the hunt while Bob shoots at 51. What is the probability that Bob hitting more ducks than Alice?

Problem 2.3. One hundred people are waiting in line at a movie theater for yet another Avengers' movies. Fifty of them have 10-dollar bills and the other 50 people have only 20-dollar bills. The tickets cost 10 dollars each. When the movie theater opens there is no money in the cashier. If each customer buys just one ticket, what is the probability that none of them will have to wait for change?

Problem 2.4. What is the probability that the cube of an integer selected at random will end with the digit 11?

3. Theory of Numbers

Problem 3.1. Find a four digit number which is a square of an integer, and such that its first two digits are the same and also its last two digits are the same.

Problem 3.2. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (that is, 1, 16, 31, etc.). This process is continued until a number is reached which has already been marked. How many unmarked numbers remain?

Problem 3.3. If the integers from 1 to 222,222,222 are written down in succession, how many 0's are written?

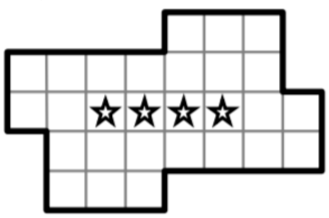
Problem 3.4. What is the next number in the following sequence?

$$4,\ 14,\ 1114,\ 3114,\ 132114,\ 1113122114,\ 311311222114,\ 13211321322114.$$

4. Miscellaneous Problems

Problem 4.1. Show that every polynomial that takes non-negative values is actually the sum of squares of some other polynomials.

Problem 4.2. Cut this figure into 4 parts, so that all parts are equal and each has a star in it.



Problem 4.3. Let $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when $f(x^{12})$ is divided by $f(x)$?

Problem 4.4. Find all positive integer solutions to $x^2 + 3 = y(x + 2)$.

Problem 4.5. Find all polynomials $p(n)$ with degree less than or equal to 3 that has rational coefficients and has integer values when n is an integer. For example, one such polynomial is

$$p(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \;=\; \frac{n(n+1)(2n+1)}{6} \;=\; 1^2 + 2^2 + ... + n^2,$$

which has rational coefficients and integer values for all integer n.

Problem 4.6. Compute

$$1 \cdot \binom{n}{1} + 3 \cdot \binom{n}{3} + 5 \cdot \binom{n}{5} + \cdots$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. (Note: ask an instructor for what a binomial coefficient is if needed.) You may assume that $n \geq 3$.

Problem 4.7. Compute the following sum:

$$\binom{100}{0}^2 + \binom{100}{1}^2 + \binom{100}{2}^2 + \ldots + \binom{100}{100}^2.$$

Problem 4.8. Let $\lfloor x \rfloor$ represents the largest integer that doesn't exceed x, e.g. $\lfloor 1.5 \rfloor = 1$, $\lfloor -1.3 \rfloor = -2$. Let $f(x) = \lfloor x \times \lfloor x \rfloor \rfloor$, e.g. $f(2.4) = \lfloor 2.4 \times \lfloor 2.4 \rfloor \rfloor = \lfloor 2.4 \times 2 \rfloor = \lfloor 4.8 \rfloor = 4$. For any integer $n \geq 1$, let a_n be the number of distinct integers in the set $\{f(x) : 0 \leq x < n\}$. Find the value of n for which $\frac{a_n+97}{n}$ is minimized.