WINTER 2022

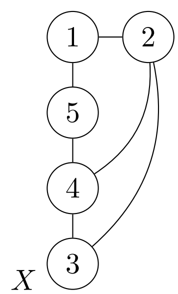
OLGA RADKO MATH CIRCLE
ADVANCED 2
FEBRUARY 7, 2021

1. Spanning trees

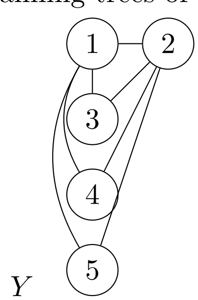
Problem 1.1. Let T be a tree with n vertices.

(a) What is the maximum number of leaves T can have?
(b) How many distinct labeled trees with n vertices are there with that many leaves?

Problem 1.2. Recall that a spanning tree of a graph G can be created by keeping all the vertices of G and deleting some edges so that what's left is a tree. Find the number of spanning trees of the following graph:

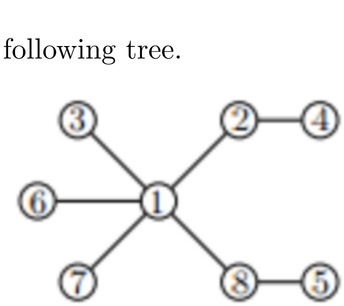


Problem 1.3. Find the number of spanning trees of the following graph:



Problem 1.4. Recall the following definition of the Prufer code of a tree T: Start with an empty list. Then, keep removing the leaf with the smallest label until two vertices remain. Each time a leaf is removed, add its neighbor to the list. The resulting length $n-2$ sequence is the Prufer code of T.

Find the Prufer code for the following tree.



Problem 1.5. Build the tree that has Prufer code (3,1,3,3,1,6).

2. Peano Axiom

Problem 2.1. Recall that we defined $0 = \emptyset$ and $S(x) = x \cup \{x\}$. Then we defined $1 = S(0)$, $2 = S(1)$, etc. Say whether the following are true or false. All must be correct for points.

(1) $3 \subseteq 5$
(2) $3 \in 5$
(3) $2 = \{\{\emptyset\}\}$

Problem 2.2. We defined Peano addition and Peano multiplication for Peano natural numbers. Give a suitable definition for Peano exponentiation of Peano natural numbers: that is, how should one define m^n?

3. Probability

Problem 3.1. You choose exactly one of these two games to play with an instructor:

- A number from 1 to 10 is selected at random. You gain full score of this problem if the number is even, and you gain no point otherwise.
- You play two round of the following game: A number from 1 to 10 is selected at random, and you win the round if the number is divisible by 3. If you win the first round, then you gain full score of this problem. If you lose the first round but win the second round, you gain one half of the score of this problem. If you lose both rounds then you gain no points.

Problem 3.2. Two hunters Alice and Bob set out to hunt ducks. Each of them hits as often as they miss when shooting at ducks. Alice shoots at 50 ducks during the hunt while Bob shoots at 51. What is the probability that Bob hits more ducks than Alice?

Problem 3.3. One hundred people are waiting in line at a movie theater for yet another Avengers' movie. Fifty of them have 10-dollar bills and the other 50 people have only 20-dollar bills. The tickets cost 10 dollars each. When the movie theater opens there is no money in the cashier. If each customer buys just one ticket, what is the probability that none of them will have to wait for change?

Problem 3.4. What is the probability that the square of an integer selected at random will end with the digit 1?

Problem 3.5. What is the probability that the cube of an integer selected at random will end with the digit 11?

4. Problems of Numbers

Problem 4.1. Find a four digit number which is a square of an integer, and such that its first two digits are the same and also its last two digits are the same.

Problem 4.2. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (that is, 1, 16, 31, etc.). This process is continued until a number is reached which has already been marked. How many unmarked numbers remain?

Problem 4.3. If the integers from 1 to 222,222,222 are written down in succession, how many 0's are written?

Problem 4.4. What is the next number in the following sequence?

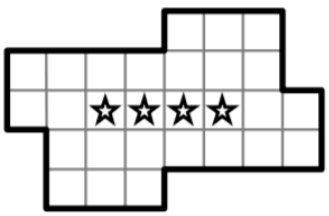
$$4,\ 14,\ 1114,\ 3114,\ 132114,\ 1113122114,\ 311311222114,\ 13211321322114.$$

5. Miscellaneous

Problem 5.1. The tech group in the a company consists of 12 people, where 3 of those are tech leads. How many ways are there to split those 12 people into 3 groups consisting of 3, 4 and 5 people if each team needs to have one tech lead?

Problem 5.2. Given a rectangle $ABCD$ with point A on the bottom-left corner and A, B, C, D arranges counter-clockwisely. $AB = 6$ units length and $AD = 5$ units length. A tortoise can either walk upward or rightward for 1 unit length for each step. How many different ways can the tortoise walk from A to C?

Problem 5.3. Cut this figure into 4 parts along the gridlines, so that all parts are equal and each has a star in it.



Problem 5.4. Compute the following sum:

$$\binom{100}{0}^2 + \binom{100}{1}^2 + \binom{100}{2}^2 + \cdots + \binom{100}{100}^2,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. (Note: ask an instructor for what a binomial coefficient is if needed.)

Problem 5.5. Find

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 7n}.$$

Problem 5.6. Let $a_0 = 1$, and define $a_{n+1} = \sqrt{a_n + 1}$ recursively. What does a_n converge to? (If you know limits, this is equivalent to asking $\lim_{n\to\infty} a_n$).

Problem 5.7. Let $\lfloor x \rfloor$ represents the largest integer that doesn't exceed x, e.g. $\lfloor 1.5 \rfloor = 1$, $\lfloor -1.3 \rfloor = -2$. Let $f(x) = \lfloor x \times \lfloor x \rfloor \rfloor$, e.g. $f(2.4) = \lfloor 2.4 \times \lfloor 2.4 \rfloor \rfloor = \lfloor 2.4 \times 2 \rfloor = \lfloor 4.8 \rfloor = 4$.

For any integer $n \geq 1$, let a_n be the number of distinct integers in $f(1)$, $f(2)$, …, $f(n)$. Find the value of n for which $\frac{a_n + 97}{n}$ is minimized.