Polynomials I - The Cubic Formula

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Adapted from worksheets by Oleg Gleizer.

1 Cubic Equations by Long Division

Definition 1 *A **cubic polynomial** (cubic for short) is a polynomial of the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$.*

The *Fundamental Theorem of Algebra* (which we will not prove this week) tells us that all cubics have three roots in the complex numbers. Recall:

Definition 2

- *The **rectangular form** of a complex number is $a + bi$, where a is the **real part** and b (not bi!) is the **imaginary part**.*
- *The **polar form** of a complex number is $re^{i\theta}$, where r is the **modulus** and θ is the **argument**. Note that two arguments which differ by an integer multiple of 2π give the same complex number.*
- *These two forms of complex numbers are related by:*

$$a = r\cos(\theta) \text{ and } b = r\sin(\theta)$$
$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

The first method of solving cubics will be for the simplest case. As an example, let's see how we would divide $x^3 + 3x^2 + 5x - 4$ by $x - 1$. Similarly to the usual long division, we multiply the *divisor* by the most simple thing (a monomial) such that when we subtract the result, the *leading terms* of the polynomial cancel. We then repeat until we get a *quotient* and a *remainder*, like so:

$$\begin{array}{r l}
 & \quad\ \, x^2 + 4x + 9 \\ \cline{2-2}
x - 1 \big) & \ x^3 + 3x^2 + 5x - 4 \\
 & -x^3 + x^2 \\ \cline{2-2}
 & \qquad 4x^2 + 5x \\
 & \quad\ -4x^2 + 4x \\ \cline{2-2}
 & \qquad\qquad 9x - 4 \\
 & \qquad\quad\ -9x + 9 \\ \cline{2-2}
 & \qquad\qquad\qquad 5
\end{array}$$

In this case, we obtain a *quotient* of $x^2 + 4x + 9$ and a *remainder* of 5 - in general, the remainder is always lower degree than the divisor. In order to use this method to solve cubics, we will need to first **find one root** of our cubic. Once we have found one root (say, r), there will be no remainder after long dividing by $x - r$. To see this, let's work through an example.

Problem 1 *Let's find all the roots of* $x^3 - 5x - 2$.

- Find one root of $x^3 - 5x - 2$. (Hint: Guess and check - it's an integer.)

- Perform long division to divide $x^3 - 5x - 2$ by $x - r$, where r is the root you found.

- Find the other two roots. (Hint: You should have gotten a quadratic from the previous part. How does one solve a quadratic?)

The main flaw of this method is that it requires us to be able to look at a cubic and guess one of its roots. Unfortunately, that's not always so easy. Thankfully, we have another method to find one root, from Tartaglia (but named after Cardano - long story!)

2 Depressed Cubics and Tartaglia's Method

Definition 3 *A cubic is said to be in* ***depressed form*** *if its leading coefficient is* 1 *and its second coefficient is* 0. *In other words, a cubic in depressed form is written* $x^3 + px + q$.

Problem 2 *Show that any cubic can be brought to depressed form.*

- Start with a general cubic $ax^3 + bx^2 + cx + d$. Why can we assume that $a = 1$?

- In order to depress the cubic $x^3 + bx^2 + cx + d$, we make the *substitution* $x = y - \alpha$. Find the correct value for α to make the resulting cubic depressed.

Problem 3 *Given a depressed cubic $x^3 + px + q$:*

- Make the substitution $x = u + v$, and expand out everything.

- In order to find the roots, let us set the above equation equal to zero. Try factoring out a $u + v$ (but in a way such that you don't just get your original equation back!) and write down some relationships between u, v, p, q.

- Using the relations we found above, rewrite the equation in terms of only u. Making the substitution $y = u^3$, solve the resulting equation for u.

- Solve for v and write down a root of $x^3 + px + q$. (The correct answer is on the next page for your reference.)

We should have seen that

Theorem 1 *(Cardano's Formula) Given a depressed cubic $x^3 + px + q$, one of its roots is given by*

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Problem 4 *Use Cardano's Formula to find one root of the following polynomials.*

- $x^3 + 6x - 2$

- $x^3 + 6x^2 + 9x - 2$ (Hint: You will need to bring this to the depressed form first.)

Problem 5 *Use Cardano's Formula to find a root of the polynomial $x^3 - 5x - 2$ from earlier. Which root did you find?*

3 Roots of Unity and the General Cubic Formula

As you may have noticed, Cardano's Formula is a very inconvenient way to solve a cubic, as you will now need to divide the cubic by $x - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. That's usually not an easy task, but thankfully our study of complex numbers (recall from last quarter!) will help us simplify the process.

Problem 6

- *What are **all** arguments of* 1*?*

- *Find **all*** n^{th} *roots (in polar form) of* 1*, ie **every** complex* z *such that* $z^n = 1$*. These are called the* n^{th} ***roots of unity**.*

- *Graph the* 3^{rd} *roots of unity in the complex plane. Then the* 4^{th}*,* 5^{th}*, etc. until you see a pattern. What pattern do you notice?*

Definition 4 *An n^{th} root of unity z is called **primitive** if $z^m \neq 1$ for all $m = 1, ..., n-1$. Primitive roots of unity are usually denoted ζ.*

Problem 7 *Prove that an n^{th} root of unith $e^{2\pi i k/n}$ is primitive if and only if k and n are relatively prime.*

In the case of cubics, 1 and 2 are both relatively prime to 3, so we can pick either corresponding root of unity to be ζ.

Problem 8 *Write ζ (either one) in rectangular form.*

Problem 9 *In terms of ζ, find all three solutions to $x^3 = c$, for any real number c.*

Problem 10 • *Using Problem 9, return to Problem 3. In the step where you solved for u, solve for* ***all three*** *possible values of u.*

- *Prove that for a depressed cubic $x^3 + px + q$,* ***all three*** *of its roots are given by*

$$x_1 = u + v, x_2 = \zeta u + \zeta^2 v, x_3 = \zeta^2 u + \zeta v$$
$$\text{where } u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \text{ and } v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

This is the ***general cubic formula****.*

Problem 11 *Use the formula from Problem 10 to find all three roots of the following polynomials.*

- $x^3 + 6x - 2$

- $x^3 + 6x^2 + 9x - 2$ (Hint: You will need to bring this to the depressed form first.)

Problem 12 *Let's return to our first example, $x^3 - 5x - 2$. Use the formula you derived in Problem 10. Which root is x_1? x_2? x_3?*

4 Bonus Section: Discriminants of Cubics

In the case of quadratics, the *discriminant* $b^2 - 4ac$ determines whether the two roots are both real or not. There is a generalized version for any kind of polynomial.

Definition 5 *Let $r_1, ..., r_n$ be the roots of a degree n polynomial $p(x) = a_n x^n + ... + a_0$. Then the discriminant of p is given by*

$$\Delta = a_n^{2n-2} \prod_{i<j} (r_i - r_j)^2$$

Problem 13 *In terms of the three roots r_1, r_2, r_3, give the formula for the discriminant of a cubic.*

Let's classify cubics based on their discriminants.

Problem 14 *Suppose that a complex number z is a root of a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$. Show that its conjugate $\overline{z}$ is also a root of p.*

Problem 15 *Given a cubic p and its discriminant Δ, prove that:*

- *If $\Delta = 0$, then p has a repeated root.*

- *If $\Delta > 0$, then p has three real roots.*

- *If $\Delta < 0$, then p has one real root and two non-real roots.*

Problem 16 *For each cubic below, is its discriminant positive, negative, or zero?*

- $x^3 - 5x - 2$

- $x^3 + 6x - 2$

- $x^3 - 3x^2 + 3x - 1$