Polynomials I - The Cubic Formula

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Adapted from worksheets by Oleg Gleizer.

1 Cubic Equations by Long Division

Definition 1 A cubic polynomial (cubic for short) is a polynomial of the form \( ax^3 + bx^2 + cx + d \), where \( a \neq 0 \).

The Fundamental Theorem of Algebra (which we will not prove this week) tells us that all cubics have three roots in the complex numbers. Recall:

Definition 2
- The rectangular form of a complex number is \( a + bi \), where \( a \) is the real part and \( b \) (not \( bi! \)) is the imaginary part.
- The polar form of a complex number is \( re^{i\theta} \), where \( r \) is the modulus and \( \theta \) is the argument. Note that two arguments which differ by an integer multiple of \( 2\pi \) give the same complex number.
- These two forms of complex numbers are related by:
  \[
  a = r \cos(\theta) \quad \text{and} \quad b = r \sin(\theta)
  \]
  \[
  r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)
  \]

The first method of solving cubics will be for the simplest case. As an example, let’s see how we would divide \( x^3 + 3x^2 + 5x - 4 \) by \( x - 1 \). Similarly to the usual long division, we multiply the divisor by the most simple thing (a monomial) such that when we subtract the result, the leading terms of the polynomial cancel. We then repeat until we get a quotient and a remainder, like so:

\[
\begin{array}{c}
  x - 1 \\
  \overline{x^3 + 3x^2 + 5x - 4} \\
  - x^3 - x^2 \\
  \hline
  4x^2 + 5x \\
  - 4x^2 + 4x \\
  \hline
  9x - 4 \\
  - 9x + 9 \\
  \hline
  5
\end{array}
\]

In this case, we obtain a quotient of \( x^2 + 4x + 9 \) and a remainder of 5 - in general, the remainder is always lower degree than the divisor. In order to use this method to solve cubics, we will need to first find one root of our cubic. Once we have found one root (say, \( r \)), there will be no remainder after long dividing by \( x - r \). To see this, let’s work through an example.
Problem 1 Let’s find all the roots of $x^3 - 5x - 2$.

- Find one root of $x^3 - 5x - 2$. (Hint: Guess and check - it’s an integer.)

- Perform long division to divide $x^3 - 5x - 2$ by $x - r$, where $r$ is the root you found.

- Find the other two roots. (Hint: You should have gotten a quadratic from the previous part. How does one solve a quadratic?)

The main flaw of this method is that it requires us to be able to look at a cubic and guess one of its roots. Unfortunately, that’s not always so easy. Thankfully, we have another method to find one root, from Tartaglia (but named after Cardano - long story!)

2 Depressed Cubics and Tartaglia’s Method

Definition 3 A cubic is said to be in depressed form if its leading coefficient is 1 and its second coefficient is 0. In other words, a cubic in depressed form is written $x^3 + px + q$.

Problem 2 Show that any cubic can be brought to depressed form.

- Start with a general cubic $ax^3 + bx^2 + cx + d$. Why can we assume that $a = 1$?

- In order to depress the cubic $x^3 + bx^2 + cx + d$, we make the substitution $x = y - \alpha$. Find the correct value for $\alpha$ to make the resulting cubic depressed.
Problem 3  Given a depressed cubic $x^3 + px + q$:

- Make the substitution $x = u + v$, and expand out everything.

- In order to find the roots, let us set the above equation equal to zero. Try factoring out a $u + v$ (but in a way such that you don’t just get your original equation back!) and write down some relationships between $u, v, p, q$.

- Using the relations we found above, rewrite the equation in terms of only $u$. Making the substitution $y = u^3$, solve the resulting equation for $u$.

- Solve for $v$ and write down a root of $x^3 + px + q$. (The correct answer is on the next page for your reference.)
We should have seen that

**Theorem 1 (Cardano’s Formula)** Given a depressed cubic \( x^3 + px + q \), one of its roots is given by

\[
\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}
\]

**Problem 4** Use Cardano’s Formula to find one root of the following polynomials.

- \( x^3 + 6x - 2 \)
- \( x^3 + 6x^2 + 9x - 2 \) (Hint: You will need to bring this to the depressed form first.)

**Problem 5** Use Cardano’s Formula to find a root of the polynomial \( x^3 - 5x - 2 \) from earlier. Which root did you find?
3 Roots of Unity and the General Cubic Formula

As you may have noticed, Cardano’s Formula is a very inconvenient way to solve a cubic, as you will now need to divide the cubic by $x - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} - \frac{9}{4} \cdot \frac{p}{3}}$. That’s usually not an easy task, but thankfully our study of complex numbers (recall from last quarter!) will help us simplify the process.

Problem 6  

• What are all arguments of 1?

• Find all $n^{th}$ roots (in polar form) of 1, ie every complex $z$ such that $z^n = 1$. These are called the $n^{th}$ roots of unity.

• Graph the 3$^{rd}$ roots of unity in the complex plane. Then the 4$^{th}$, 5$^{th}$, etc. until you see a pattern. What pattern do you notice?
Definition 4 An $n$th root of unity $z$ is called primitive if $z^m \neq 1$ for all $m = 1, \ldots, n - 1$. Primitive roots of unity are usually denoted $\zeta$.

Problem 7 Prove that an $n$th root of unity $e^{2\pi ik/n}$ is primitive if and only if $k$ and $n$ are relatively prime.

In the case of cubics, 1 and 2 are both relatively prime to 3, so we can pick either corresponding root of unity to be $\zeta$.

Problem 8 Write $\zeta$ (either one) in rectangular form.

Problem 9 In terms of $\zeta$, find all three solutions to $x^3 = c$, for any real number $c$. 
Problem 10  
• Using Problem 9, return to Problem 3. In the step where you solved for $u$, solve for all three possible values of $u$.

• Prove that for a depressed cubic $x^3 + px + q$, all three of its roots are given by

$$x_1 = u + v, x_2 = \zeta u + \zeta^2 v, x_3 = \zeta^2 u + \zeta v$$

where $u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \text{and} \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

This is the general cubic formula.
Problem 11 Use the formula from Problem 10 to find all three roots of the following polynomials.

- $x^3 + 6x - 2$

- $x^3 + 6x^2 + 9x - 2$ (Hint: You will need to bring this to the depressed form first.)

Problem 12 Let’s return to our first example, $x^3 - 5x - 2$. Use the formula you derived in Problem 10. Which root is $x_1$? $x_2$? $x_3$?
4 Bonus Section: Discriminants of Cubics

In the case of quadratics, the discriminant \( b^2 - 4ac \) determines whether the two roots are both real or not. There is a generalized version for any kind of polynomial.

**Definition 5** Let \( r_1, ..., r_n \) be the roots of a degree \( n \) polynomial \( p(x) = a_n x^n + ... + a_0 \). Then the discriminant of \( p \) is given by

\[
\Delta = a_n^{2n-2} \prod_{i<j} (r_i - r_j)^2
\]

**Problem 13** In terms of the three roots \( r_1, r_2, r_3 \), give the formula for the discriminant of a cubic.

Let’s classify cubics based on their discriminants.

**Problem 14** Suppose that a complex number \( z \) is a root of a cubic polynomial \( p(x) = ax^3 + bx^2 + cx + d \). Show that its conjugate \( \overline{z} \) is also a root of \( p \).
Problem 15  Given a cubic $p$ and its discriminant $\Delta$, prove that:

- If $\Delta = 0$, then $p$ has a repeated root.

- If $\Delta > 0$, then $p$ has three real roots.

- If $\Delta < 0$, then $p$ has one real root and two non-real roots.

Problem 16  For each cubic below, is its discriminant positive, negative, or zero?

- $x^3 - 5x - 2$

- $x^3 + 6x - 2$

- $x^3 - 3x^2 + 3x - 1$