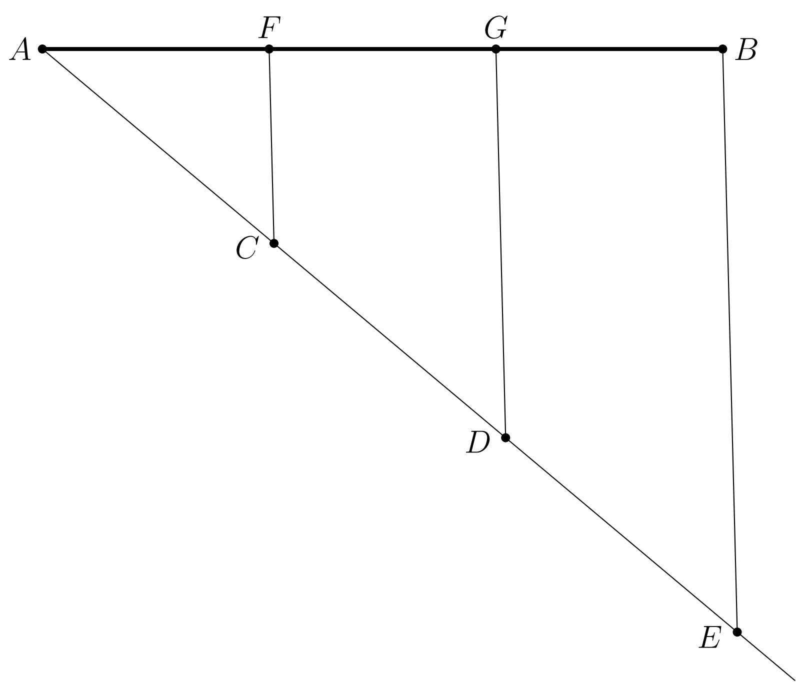
Vector Geometry Lesson 1

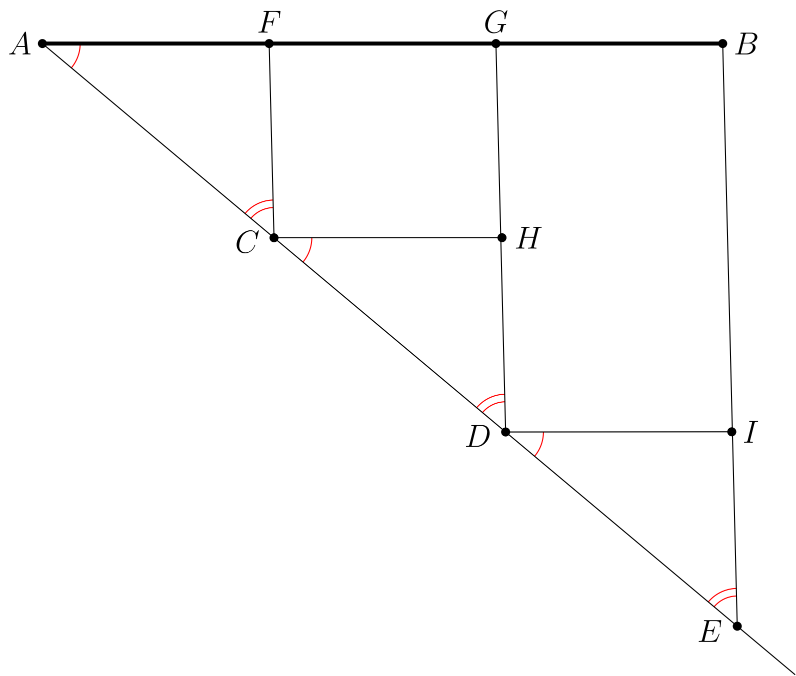
How to divide a segment into parts

To divide a given segment AB into three equal parts, let us draw a ray having one of the segment's vertices, point A on the picture below, as a vertex of its own. Let us further mark three segments, AC, CD, and DE, of equal length on the ray. Let us connect points B and E by a straight line segment. Let us further construct straight lines parallel to BE and passing through the points C and D. The points F and G where the lines meet the line AB will divide the segment AB into three parts of equal length. Similarly, one can divide a segment into any (positive integral) number of parts.



Proposition 1 $|AF| = |FG| = |GB|$

Proof — Draw straight lines CH and DI parallel to AB.



Problem 1 *Complete the proof of Proposition 1 by filling the Reason part of the Claim-Reason chart below (continues on the next page).*

Claim	**Reason**
$\lvert AC\rvert = \lvert CD\rvert = \lvert DE\rvert$	

Claim	**Reason**
$\angle FAC \cong \angle HCD \cong \angle IDE$	
$\angle FCA \cong \angle HDC \cong \angle IED$	
$\triangle ACF \cong \triangle CDH \cong \triangle DEI$	
$\lvert AF\rvert = \lvert CH\rvert = \lvert DI\rvert$	
$CHGF$ *and* $DIBG$	

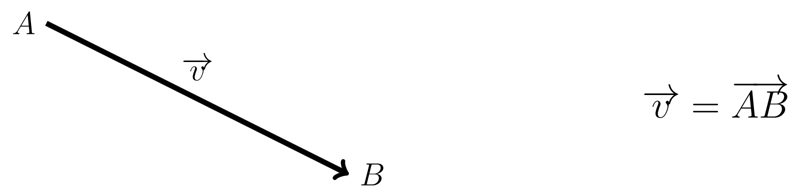
Claim	**Reason**
$\lvert CH\rvert = \lvert FG\rvert$	
$\lvert DI\rvert = \lvert GB\rvert$	
$\lvert AF\rvert = \lvert FG\rvert = \lvert GB\rvert$	

Q.E.D.

Problem 2 *Use a compass and a ruler to divide the segment below into five parts of equal length.*

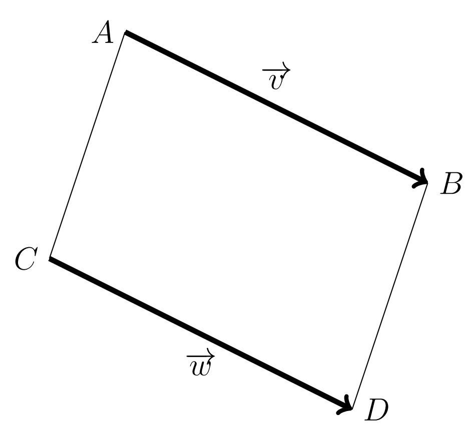
Vectors in the Euclidean plane

Definition 1 *A vector in the Euclidean plane is a directed segment or arrow.*



For the vector $\vec{v} = \overrightarrow{AB}$, point A is called *initial* and point B is called *terminal*.

Two vectors, $\vec{v} = \overrightarrow{AB}$ and $\vec{w} = \overrightarrow{CD}$ are considered equivalent if the quadrilateral $ABDC$ is a parallelogram.

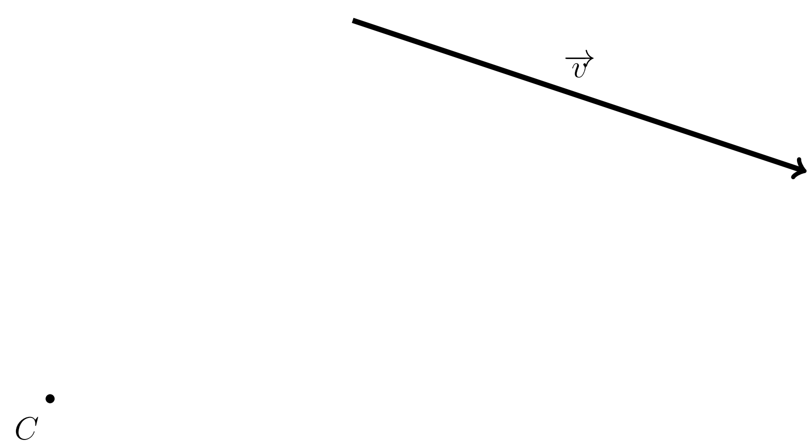


In other words, two vectors are considered the same, if the straight lines they belong to are parallel and so are the straight lines connecting their initial and terminal points. In this case, we write

$$\overrightarrow{v} = \overrightarrow{w}.$$

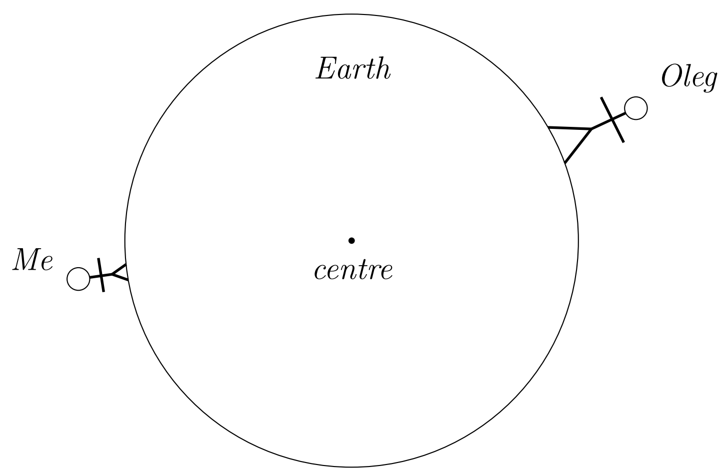
Note 1 *A vector is characterized by its direction and length. One cannot make a formal definition out of this observation, because a direction is formally defined by means of an appropriate vector.*

Problem 3 *Use a compass and a ruler to construct the vector* $\overrightarrow{w}$ *equal to the vector* $\overrightarrow{v}$ *given below and having the given point* C *as its initial point.*



Physical forces, such as the force of gravity or the electric force that pulls together two objects having a different electric charge and pushes away two objects having the same electric charge, are vectors in 3D. The direction of a vector shows the direction in which the corresponding force is acting. The length of the vector shows the strength of the force.

Problem 4 *On the picture below, draw the vectors of the gravitational pull the Earth exerts on you and on your Math Circle leader.*



- *Do the vectors of the gravitational pull point towards the centre of the Earth or elsewhere? Why or why not? (The problem continues on the next page.)*

• *Assuming that Oleg is twice as heavy as you are, how do you show it by means of the gravitational pull vectors on the above picture?*

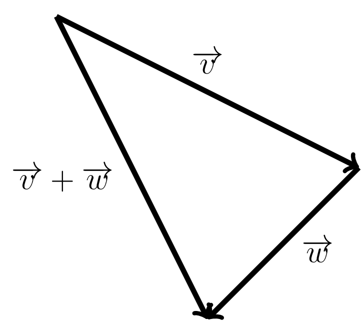
Velocities of motion are vectors, too. The direction of the velocity vector shows the direction in which the body in consideration is moving at the moment. The length of the velocity vector is called *speed*. It shows how fast the body is moving at the moment.

Problem 5 *The truck on the picture below is going 30 mph. The car on the same picture is speeding at 90 mph the opposite way. Draw the corresponding velocity vectors.*

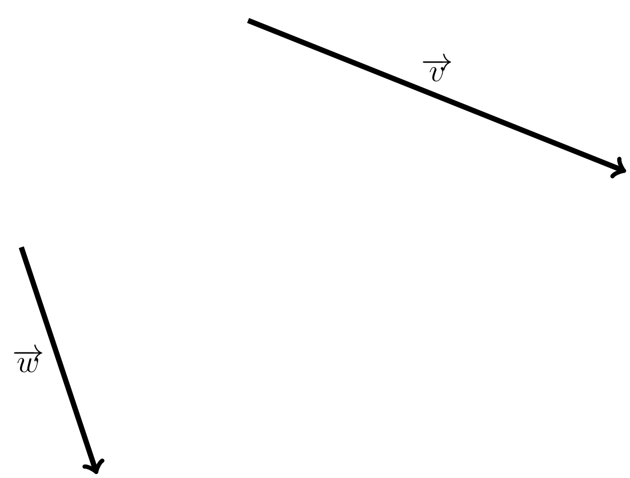
Velocities and forces of the real world are vectors in 3D. For simplicity, we begin our study of vectors from two dimensions. Everything we are going to learn about vectors in the Euclidean plane will be valid in any Euclidean space of higher dimension, three, four, etc., with the only exception for the dimension itself.

Addition of vectors

To find the sum of two vectors, $\overrightarrow{v}$ and $\overrightarrow{w}$, one needs to take $\overrightarrow{w}$ so that the initial point of $\overrightarrow{w}$ coincides with the terminal point of $\overrightarrow{v}$. The vector originating at the initial point of $\overrightarrow{v}$ and terminating at the terminal point of $\overrightarrow{w}$ is the sum $\overrightarrow{v}+\overrightarrow{w}$.



Problem 6 *Use a compass and a ruler to construct the sum* $\overrightarrow{v}+\overrightarrow{w}$ *of the vectors* $\overrightarrow{v}$ *and* $\overrightarrow{w}$ *given below.*



Zero vector

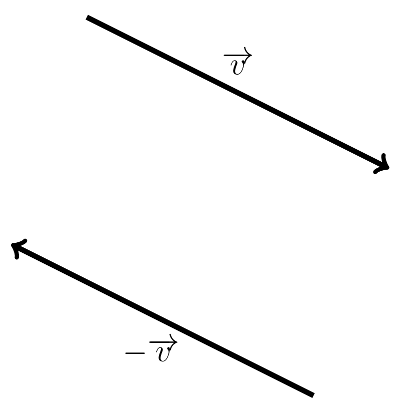
A vector that has coinciding initial and terminal points is called the *zero vector* and is denoted as $\overrightarrow{0}$. According to the above definition of the vector addition,

$$\overrightarrow{v} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{v} = \overrightarrow{v} \tag{1}$$

for any vector $\overrightarrow{v}$.

Opposite vectors

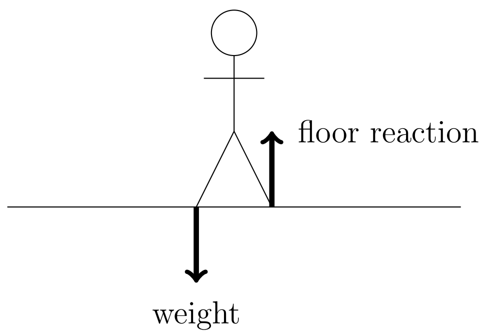
A vector $\overrightarrow{w}$ such that $\overrightarrow{w} + \overrightarrow{v} = \overrightarrow{0}$ is called *opposite* to $\overrightarrow{v}$ and is denoted as $-\overrightarrow{v}$. The vector $-\overrightarrow{v}$ lies either on the same straight line as $\overrightarrow{v}$ or on a parallel one, has the same length as $\overrightarrow{v}$, but points in the opposite direction.



Note that $-\overrightarrow{v} + \overrightarrow{v} = \overrightarrow{0}$ by definition, but the validity of the equation $\overrightarrow{v} + (-\overrightarrow{v}) = \overrightarrow{0}$ follows from the definition of vector addition given above and from the definition of the zero vector. We can combine both into the following.

$$-\overrightarrow{v} + \overrightarrow{v} = \overrightarrow{v} + (-\overrightarrow{v}) = \overrightarrow{0} \tag{2}$$

Here is an important example of an opposite vector. When you stand still, the floor pushes you up with the force opposite to the force of the gravitational pull, a.k.a. weight.



The opposite vectors add up to the zero resulting force acting on you, hence you don't move.

Problem 7 *Give an example of a different pair of opposite forces.*

The following is the last thing to mention about the opposite vectors. The formula $\overrightarrow{w} - \overrightarrow{v}$ is defined as $\overrightarrow{w} + (-\overrightarrow{v})$ for any vectors $\overrightarrow{v}$ and $\overrightarrow{w}$.

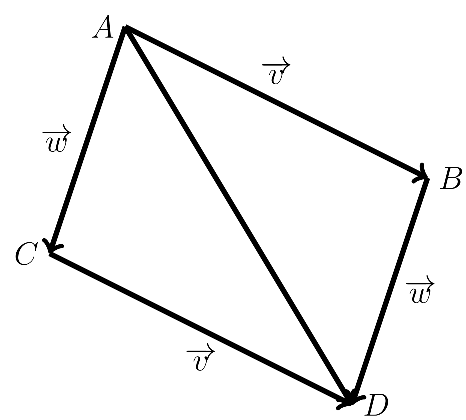
$$\overrightarrow{w} - \overrightarrow{v} = \overrightarrow{w} + (-\overrightarrow{v}) \tag{3}$$

Commutativity of vector addition

Addition of vectors is commutative. In other words, $\overrightarrow{v} + \overrightarrow{w}$ always equals to $\overrightarrow{w} + \overrightarrow{v}$.

$$\overrightarrow{v} + \overrightarrow{w} = \overrightarrow{w} + \overrightarrow{v} \tag{4}$$

Consider the below parallelogram $ABCD$.



According to our definition of a vector, $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{v}$ and $\overrightarrow{AC} = \overrightarrow{BD} = \overrightarrow{w}$. The initial point of both vectors $\overrightarrow{v} + \overrightarrow{w}$ and $\overrightarrow{w} + \overrightarrow{v}$ is A, their common terminal point is D, hence addition of vectors in the Euclidean plane is commutative.

If $\overrightarrow{w} = -\overrightarrow{v}$ or one of the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ is a zero vector, then the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ do not span a parallelogram and the above argument does not work. That is why we have proven commutativity of addition for these special cases separately, see (1) and (2).